$\mathcal{H}_2$ optimal and frequency limited approximation methods for large-scale LTI dynamical systems

Pierre Vuillemin * **Charles Poussot-Vassal** ** **Daniel Alazard** *

* *Université de Toulouse and Onera - The French Aerospace Lab, F-31055 Toulouse, France (email : pierre.vuillemin@onera.fr)*
** *Onera - The French Aerospace Lab, F-31055 Toulouse, France*

Abstract: Model order reduction over a bounded frequency range is more adapted than the standard $\mathcal{H}_2$ approximation whenever the entire frequential behaviour of the large-scale model is not needed or not accurately known. However most of the methods that enable to reduce a model on a limited frequency range are based on the use of weights. Yet their determination is often an issue for engineers. That is why, in this paper, two weight-free model approximation algorithms are proposed. They are based on recent algorithms that achieve local $\mathcal{H}_2$ optimal model reduction (see Gugercin (2007), Van Dooren et al. (2008) and Gugercin et al. (2008)). The proposed algorithms efficiency are validated both on a standard benchmark and on an industrial use case.

Keywords: LTI dynamical system, model reduction, large-scale approximation

1. INTRODUCTION

1.1 Motivations & contributions

Linear dynamical models are widely used to represent the behaviour of physical systems or phenomena. Depending on the required accuracy and on the complexity of the physical system, the size of the linear model can become large [1]. This is not a theoretical issue, but rather a practical one since this kind of models require a lot of resources to be analysed or simulated, and finite precision arithmetic may disrupt theoretical results. For instance, it is not possible - or not possible in an acceptable time - to synthesize a controller with generic tools on a large-scale model. A traditional way to handle such models consists in approximating them by smaller ones.

The approximation is performed in order to achieve a given objective. Usually, it is done such that the reduced-order model accurately reproduces the frequential behaviour of the original one *on the whole frequency range*. However, in some cases, such a constraint can appear to be too binding, indeed : *(i)* some frequencies are physically meaningless and can be viewed as uncertainties, *(ii)* in practice, actuators and sensors bandwidth are limited which make some frequencies irrelevant for control purpose and *(iii)* some frequencies are more specifically of interest, *e.g.* when vibration control has to be performed. In these cases, considering the problem of reducing the full-order model such that a good approximation is found *over a bounded frequency range* can be more appropriate and appealing for engineers.

To this aim, two new algorithms achieving frequency-limited model approximation are proposed in this paper, namely the Frequency-Limited Iterative SVD-Tangential Algorithm (FL-ISTIA) and the Frequency-Limited Two-Sided Iteration Algorithm (FL-TSIA).

1.2 Projection based model reduction framework

Within model reduction approaches, the projection framework is well appropriated. It is grounded on the Petrov-Galerkin conditions and is recalled in Problem 1.

Problem 1. Given a continuous, stable and strictly proper MIMO LTI dynamical model Σ, with simple poles, defined as

$$\Sigma := \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$ and $C \in \mathbb{R}^{n_y\times n}$, the projection-based model order reduction problem consists in finding $V, W \in \mathbb{R}^{n\times r}$ with $W^TV = I_r$ such that the reduced-order model $\hat{\Sigma}$ of order $r \ll n$ which realisation is $\left(\hat{A}, \hat{B}, \hat{C}\right)$ with $\hat{A} = W^TAV$, $\hat{B} = W^TB$ and $\hat{C} = CV$, accurately reproduces the behaviour of the full-order system Σ.

The accuracy of the reduced-order model can be measured through several norms depending on the reduction objective. In this paper the $\mathcal{H}_2$-norm and the frequency-limited $\mathcal{H}_2$-norm, denoted $\mathcal{H}_{2,\omega}$-norm [2], are considered. Two model reduction problems can hence be formulated. First, the $\mathcal{H}_2$ model reduction problem, recalled in Problem 2, which aims at reproducing on average the behaviour of a large-scale model for all frequencies.

Problem 2. Considering the setting given in Problem 1, the projection based $\mathcal{H}_2$-norm model reduction problem

[1] In this paper, a model with 1000 to 10000 states is considered as large, above, it is considered as very-large scale.

[2] Its formal definition is given in Section 3.

consists in finding the projectors $V, W \in \mathbb{R}^{n\times r}$ which enable to minimize the $\mathcal{H}_2$-norm of the error, i.e.

$$\mathcal{J}_{\mathcal{H}_2}\left(\hat{A},\hat{B},\hat{C}\right) = \|\Sigma - \hat{\Sigma}\|^2_{\mathcal{H}_2} \quad (2)$$

for a given reduced order $r \ll n$.

Secondly, the $\mathcal{H}_{2,\omega}$ model approximation problem, recalled in Problem 3, deals with the approximation of a large-scale model over a bounded frequency range.

Problem 3. Considering the setting given in Problem 1, the projection-based frequency-limited model order reduction problem consists in finding the projectors $V, W \in \mathbb{R}^{n\times r}$ which enable to minimize the $\mathcal{H}_{2,\omega}$-norm of the error, i.e.

$$\mathcal{J}_{\mathcal{H}_{2,\omega}}\left(\hat{A},\hat{B},\hat{C}\right) = \|\Sigma - \hat{\Sigma}\|^2_{\mathcal{H}_{2,\omega}} \quad (3)$$

for a given reduced order $r \ll n$.

1.3 Paper structure & notations

The paper is organised as follows. Section 2 addresses some well established methods to solve the $\mathcal{H}_2$ model reduction problem. More specifically, three efficient algorithms are presented. The frequency-limited model reduction problem is then tackled in Section 3 where frequency-limited variants of the $\mathcal{H}_2$ algorithms are proposed. In Section 4, all the presented model reduction methods are finally applied on a commonly used benchmark model and on an industrial aircraft model. Conclusion and comments are gathered in Section 5.

In this paper, entities with a hat $\hat{.}$ refers to the reduced order model, $H(s) = C(sI - A)^{-1}B$ is a transfer matrix, $\mathbf{log}\,(A)$ is the matrix logarithm of A, A^* is the conjugate transpose of A and A^T its transpose, $\mathcal{Q}$ and $\mathcal{P}$ denote respectively the observability and reachability gramians, $\mathcal{Q}_\omega$ and $\mathcal{P}_\omega$ denote their frequency-limited variants.

2. $\mathcal{H}_2$ OPTIMAL MODEL REDUCTION

2.1 Preliminary results

As Problem 2 is not convex, attention has been given to seek the corresponding first-order necessary optimality conditions. This has firstly been achieved in Wilson (1974) using the gramians of the system and in Meier and Luenberger (1967) using interpolation conditions related to the transfer functions. In Gugercin et al. (2008), by linking these conditions to the interpolatory framework set in Grimme (1997), the authors proposed an algorithm called *Iterative Rational Krylov Algorithm* (IRKA) which enable to find a local minimizer [3] for Problem 2 for the SISO case. MIMO systems can be handled with the tangential interpolatory framework introduced in Gallivan et al. (2004) for which equivalent first-order optimality conditions have been presented in Van Dooren et al. (2008). They are recalled in Theorem 1.

Theorem 1. If the gradients of $\mathcal{J}_{\mathcal{H}_2}$ with respect to $\hat{A}$, $\hat{B}$ and $\hat{C}$ respectively, are equal to zero, i.e. $\nabla_{\hat{A}}\mathcal{J}_{\mathcal{H}_2} = 0$, $\nabla_{\hat{B}}\mathcal{J}_{\mathcal{H}_2} = 0$ and $\nabla_{\hat{C}}\mathcal{J}_{\mathcal{H}_2} = 0$, then the following tangential interpolation conditions are satisfied for $i = 1,\ldots,r$:

$$\begin{aligned} [H(-\hat{\lambda}_i) - \hat{H}(-\hat{\lambda}_i)]\hat{b}_i &= 0 \\ \hat{c}_i^*[H(-\hat{\lambda}_i) - \hat{H}(-\hat{\lambda}_i)] &= 0 \\ \hat{c}_i^*\frac{d}{ds}[H(s) - \hat{H}(s)]|_{s=-\hat{\lambda}_i}\hat{b}_i &= 0 \end{aligned} \quad (4)$$

where the $\hat{\lambda}_i$ are the eigenvalues of $\hat{A}$, $\{\hat{b}_1,\ldots,\hat{b}_r\} = \hat{B}^T R$ and $\{\hat{c}_1,\ldots,\hat{c}_r\} = \hat{C}L$ (where L and R are the left and right eigenvectors associated to $\hat{\lambda}_i$).

Theorem 1 expresses the necessary conditions to find a local minimum of $\mathcal{J}_{\mathcal{H}_2}$. Hence the optimal model approximation problem consists in finding $\left\{\hat{\lambda}_i, \hat{c}_i, \hat{b}_i\right\}$ such that (4) is satisfied. Theorem 2 then makes the link with Problem 1 and shows how the projectors V and W are constructed to fulfil these conditions.

Theorem 2. Let $V \in \mathbb{C}^{n\times r}$ and $W \in \mathbb{C}^{n\times r}$ be full rank matrices such that $W^TV = I_r$. Let $\sigma_i \in \mathbb{C}^r$, $\hat{b}_i \in \mathbb{C}^{n_u}$ and $\hat{c}_i \in \mathbb{C}^{n_y}$ (for $i = 1,\ldots,r$) be given sets of interpolation points and right and left tangential directions, respectively. Assume that points σ_i are selected such that $\sigma_i I_n - A$ are invertible. If, for $i = 1,\ldots,r$,

$$\begin{aligned} (\sigma_i I_n - A)^{-1}B\hat{b}_i \in \mathbf{span}(V) \\ \textit{and } (\sigma_i I_n - A^T)^{-1}C^T\hat{c}_i \in \mathbf{span}(W) \end{aligned} \quad (5)$$

then, the reduced-order system $\hat{H}(s)$ satisfies the tangential interpolation conditions given in Theorem 1.

2.2 Iterative Tangential Interpolation Algorithm

In Van Dooren et al. (2008), an algorithm has been suggested to build V and W in accordance with Theorem 2. Here it is called *Iterative Tangential Interpolation Algorithm* (ITIA) and is presented in Algorithm 1.

The key steps are (i) the construction of Krylov subspaces (steps 1 and 8) which enables to match the moments of the original model at some frequencies $\sigma_k^{(i)}$ and (ii) the assignment of the new interpolation points (step 7) as the mirror images of the reduced-order model's poles. Note that steps 2 and 9 impose bi-orthogonality of the projectors. There are the main properties of the ITIA :

- It can be applied to large-scale models since it only involves matrices and vector operations.
- It leads to local minimum of the $\mathcal{H}_2$-norm model reduction problem.
- The quality of the approximation can strongly change depending on the initial shift selection.
- It does not guarantee the stability of the reduced-order model, even if in practice, instability is rarely observed.

As shown in Section 2.3, the ITIA can be modified in an interesting way by using a single gramian.

2.3 Iterative SVD-Tangential Interpolation Algorithm

Model reduction methods based on the *Singular Value Decomposition* (among which the balanced truncation) are numerically expensive due to the computation of two gramians, but they have some nice properties like stability

[3] In practice, the algorithm does not seem to converge towards saddle points or local extrema.

Algorithm 1 Iterative Tangential Interpolation Algorithm (ITIA)

Require: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$, $\{\sigma_1^{(0)}, \ldots, \sigma_r^{(0)}\} \in \mathbb{C}^{n\times r}$, $\{\hat{b}_1, \ldots, \hat{b}_r\} \in \mathbb{C}^{n\times r}$, $\{\hat{c}_1, \ldots, \hat{c}_r\} \in \mathbb{C}^{n\times r}$, $\varepsilon > 0$

1: Construct,

$$V = \left[(\sigma_1^{(0)} I_n - A)^{-1} B\hat{b}_1, ..., (\sigma_r^{(0)} I_n - A)^{-1} B\hat{b}_r\right]$$
$$W = \left[(\sigma_1^{(0)} I_n - A)^{-T} C^T \hat{c}_1, ..., (\sigma_r^{(0)} I_n - A)^{-T} C^T \hat{c}_r\right]$$

2: Compute $W \leftarrow W(V^T W)^{-1}$

3: **while** $|\sigma^{(i)} - \sigma^{(i-1)}| > \varepsilon$ **do**

4: $i \leftarrow i+1$, $\hat{A} = W^T AV$, $\hat{B} = W^T B$ and $\hat{C} = CV$

5: Compute $\hat{A}X = \mathbf{diag}(\lambda(\hat{A}))X$

6: Compute $\{\hat{b}_1, \ldots, \hat{b}_r\} = \hat{B}^T X$, $\{\hat{c}_1, \ldots, \hat{c}_r\} = \hat{C}X^{-1}$

7: Set $\sigma^{(i)} = -\lambda(\hat{A})$

8: Construct,

$$V = \left[(\sigma_1^{(i)} I_n - A)^{-1} B\hat{b}_1, ..., (\sigma_r^{(i)} I_n - A)^{-1} B\hat{b}_r\right]$$
$$W = \left[(\sigma_1^{(i)} I_n - A)^{-T} C^T \hat{c}_1, ..., (\sigma_r^{(i)} I_n - A)^{-T} C^T \hat{c}_r\right]$$

9: Compute $W \leftarrow W(V^T W)^{-1}$

10: **end while**

11: Construct $\hat{\Sigma} : (W^T AV, W^T B, CV)$

Ensure: $V, W \in \mathbb{R}^{n\times r}$, $W^T V = I_r$

preservation. The *Iterative SVD-Tangential Interpolation Algorithm* (ISTIA) proposed in Poussot-Vassal (2011) tries to gather the advantages of the moment-matching methods (IRKA/ITIA) and the SVD-based methods.

The basic version is the *Iterative SVD Rational Krylov Algorithm* (ISRKA) presented in Gugercin (2007). It consists in replacing one of the projectors by a new one built with one gramian. If the observability gramian $\mathcal{Q}$ is used, then the projector W is defined as :

$$W = \mathcal{Q}V(V^T\mathcal{Q}V)^{-1} \tag{6}$$

The ISRKA is applicable to SIMO or MISO models depending on which gramian is used, but for MIMO models, the tangential interpolation framework must be used, thus leading to the ISTIA. The ISTIA is similar to Algorithm 1 except that : (i) the algorithm requires the observability (resp. reachability) gramian $\mathcal{Q}$ (resp. $\mathcal{P}$) and (ii) in steps 1 and 8, W (resp. V) is now constructed in accordance with (6) (resp. $V = \mathcal{P}W(W^T\mathcal{P}W)^{-1}$). The ISTIA has different properties from the ITIA, indeed :

- Using a single gramian guarantees stability preservation (see Gugercin (2007)).
- It is usually more robust to initial shift point selection than the ITIA and converges more quickly.
- It fulfils only a subset of the optimality conditions (4).
- It requires a gramian and can thus only be used when the latter is computable.

2.4 Two-Sided Iteration Algorithm

First-order optimality conditions presented in Theorem 1 can also be achieved through the algorithm presented in Xu and Zeng (2011) and recalled in Algorithm 2. It is based on the fact that solving

$$AX + X\Gamma + BR^T = 0 \tag{7a}$$
$$A^TY + Y\Gamma + C^TL = 0 \tag{7b}$$

for X, $Y \in \mathbb{R}^{n\times r}$ with $\Gamma \in \mathbb{R}^{r\times r}$, $L \in \mathbb{R}^{n_y\times r}$ and $R \in \mathbb{R}^{r\times n_u}$, and constructing the reduced-order model such that :

$$\left(\hat{A}, \hat{B}, \hat{C}\right) = \left((X^TY)^{-1}X^TAY, (X^TY)^{-1}X^TB, CY\right) \tag{8}$$

is equivalent to a tangential interpolation of $H(s)$ and $\hat{H}(s)$ at $-\gamma_i$ (the eigenvalues of Γ) where the columns of L (resp. R) are the left (resp. right) tangential directions (see Gallivan et al. (2004) for more details).

Algorithm 2 Two Sided Iteration Algorithm (TSIA)

Require: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$, $N \in \mathbb{N}^*$, $r \in \mathbb{N}^*$ and W_0, $V_0 \in \mathbb{R}^{n\times r}$ such that $W_0^T V_0 = I_r$

1: Compute $\hat{A}_0 = W_0^T AV_0$, $\hat{B}_0 = W_0^T B$ and $\hat{C}_0 = CV_0$

2: Set $k = 0$

3: **while** $k \leq N-1$ **do**

4: Compute X_k and Y_k by solving the following Sylvester equations :

$$AX_k + X_k\hat{A}_k^T + B\hat{B}_k^T = 0$$
$$A^TY_k + Y_k\hat{A}_k - C^T\hat{C}_k = 0$$

5: Compute $W_{k+1} = Y_k(X_k^TY_k)^{-1}$ and $V_{k+1} = X_k$

6: Compute $\hat{A}_{k+1} = W_{k+1}^T AV_{k+1}$, $\hat{B}_{k+1} = W_{k+1}^T B$ and $\hat{C}_{k+1} = CV_{k+1}$

7: Set $k = k+1$

8: **end while**

9: Set $\hat{\Sigma} : (\hat{A}_N, \hat{B}_N, \hat{C}_N)$

With reference to this algorithm, it it worth noticing that :

- It can be applied on large models, as illustrated in Benner et al. (2011).
- The TSIA requires good initial projectors W_0, V_0 to be efficient. For instance the projectors obtained through another method can be used. In practice it often enables to enhance the result achieved by the ITIA or the ISTIA.
- In this form, the algorithm stops after N iterations. A better alternative consists in computing the $\mathcal{H}_2$-norm of the error at each iteration and then to keep the best solution. Indeed, considering the following expression of the $\mathcal{H}_2$ error :

$$\begin{aligned}\mathcal{J}_{\mathcal{H}_2} &= \|H(s)\|^2_{\mathcal{H}_2} - \mathbf{trace}\left(2CX\hat{C}^T\right) + \|\hat{H}(s)\|^2_{\mathcal{H}_2} \\ &= \|H(s)\|^2_{\mathcal{H}_2} + \mathbf{trace}\left(2B^TY\hat{B}\right) + \|\hat{H}(s)\|^2_{\mathcal{H}_2}\end{aligned} \tag{9}$$

  if X (or Y) is available, the computation of $\mathcal{J}_{\mathcal{H}_2} - \|H(s)\|^2_{\mathcal{H}_2}$ (the $\mathcal{H}_2$ error, up to a constant) can be achieved with little extra cost since it only requires to evaluate the $\mathcal{H}_2$-norm of a low-order model $\|\hat{H}(s)\|^2_{\mathcal{H}_2}$. At step 4 of Algorithm 2, X and Y are computed, hence only the $\mathcal{H}_2$-norm of $\hat{\Sigma}_k : \left(\hat{A}_k, \hat{B}_k, \hat{C}_k\right)$ has to be evaluated at each iteration to obtain an indication about the error which is numerically very cheap.

3. MAIN RESULTS : $\mathcal{H}_{2,\omega}$ ORIENTED MODEL REDUCTION

3.1 Preliminary results

A widely used approach to tackle the issue of model reduction over a bounded frequency range consists in applying input and/or output filters to the full-order model and to reduce it. Despite interesting results, see for instance Gugercin and Antoulas (2004), Beattie and Gugercin (2011) and Anic et al. (2012), weights determination can still be a time consuming and challenging task for engineers. That is why weight-free approaches are preferred here. They are mainly based on the frequency-limited gramians presented in Gawronski and Juang (1990) and recalled in Definition 1.

Definition 1. Given the realisation (A, B, C) of a strictly proper and stable MIMO LTI dynamical system, the frequency-limited reachability and observability gramians noted $\mathcal{P}_\omega$ and $\mathcal{Q}_\omega$, respectively, are given by :

$$\mathcal{P}_\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega} T(\nu)BB^T T^*(\nu)d\nu \qquad (10a)$$

$$\mathcal{Q}_\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega} T^*(\nu)C^T CT(\nu)d\nu \qquad (10b)$$

with $T(\nu) = (j\nu I_n - A)^{-1}$.

These gramians are solutions of the two following Lyapunov equations :

$$A\mathcal{P}_\omega + \mathcal{P}_\omega A^T + W_c(\omega) = 0 \qquad (11a)$$

$$A^T\mathcal{Q}_\omega + \mathcal{Q}_\omega A + W_o(\omega) = 0 \qquad (11b)$$

where

$$W_c(\omega) = S(\omega)BB^T + BB^T S^*(\omega) \qquad (12a)$$

$$W_o(\omega) = S^*(\omega)C^T C + C^T CS(\omega) \qquad (12b)$$

and

$$S(\omega) = \frac{j}{2\pi}\mathbf{log}\left((A + j\omega I_n)(A - j\omega I_n)^{-1}\right) \qquad (13)$$

These gramians can be used to express the restriction of the $\mathcal{H}_2$-norm on the frequency range $[0, \omega]$, see Definition 2. This expression can be found in Masi et al. (2010) and is useful to measure the accuracy of the approximation over a bounded frequency range.

Definition 2. Given a stable and strictly proper MIMO linear dynamical system Σ with $H(s) = C(sI_n - A)^{-1}B$, the $\mathcal{H}_{2,\omega}$-norm is defined as follow

$$\begin{aligned}\|H(s)\|^2_{\mathcal{H}_{2,\omega}} &= \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{trace}\left(H(j\nu)H(-j\nu)^T\right)d\nu \\ &= \mathbf{trace}\left(C\mathcal{P}_\omega C^T\right) \\ &= \mathbf{trace}\left(B^T\mathcal{Q}_\omega B\right)\end{aligned} \qquad (14)$$

where $\mathcal{P}_\omega$ and $\mathcal{Q}_\omega$ are the frequency-limited gramians.

Remark 1. The frequency-limited gramians can easily be expressed on the frequency interval $[\omega_1, \omega_2]$, indeed $\mathcal{Q}_{[\omega_1,\omega_2]} = \mathcal{Q}_{\omega_2} - \mathcal{Q}_{\omega_1}$ and $\mathcal{P}_{[\omega_1,\omega_2]} = \mathcal{P}_{\omega_2} - \mathcal{P}_{\omega_1}$. Hence a restriction of the $\mathcal{H}_2$-norm on $[\omega_1, \omega_2]$ can be formulated as well. In this article, only the interval $[0, \omega]$ is considered though.

The frequency-limited gramians can be directly used for model reduction through the *Frequency-Limited Balanced Truncation*, noted FL-BT. This method has been proposed in Gawronski and Juang (1990). It consists in using frequency-limited gramians instead of classical ones to perform a balanced truncation (see Antoulas (2005) for more details). The properties of this method are summarized thereafter :

- As shown in Gugercin and Antoulas (2004), using this approach is equivalent to apply a frequency-weighted balanced truncation with perfect filters. And indeed, the method is efficient in practice.
- Unlike the balanced truncation, it does not guarantee the stability of the reduced-order model.
- Computing the frequency-limited gramians require to solve two large-scale Lyapunov equations and to evaluate the logarithm of a large-scale matrix. This makes the frequency-limited balanced truncation numerically more complex to achieve than the basic balanced truncation.

Remark 2. First-order optimality conditions have recently been derived from the state-space formulation of the $\mathcal{H}_{2,\omega}$-norm in Petersson and Löfberg (2012). However they cannot be used as easily as those presented in Theorem 1.

3.2 Frequency Limited ISTIA

In a similar way, using one frequency-limited gramian in the ISTIA instead of the standard one makes the algorithm more efficient in terms of $\mathcal{H}_{2,\omega}$-norm. This effect can be accentuated by choosing the initial interpolation points so that their modulus belay to the concerned frequency interval. The *Frequency-Limited ISTIA* (FL-ISTIA), firstly proposed in Vuillemin et al. (2012), has the following properties :

- It leads to good results in terms of$\mathcal{H}_{2,\omega}$-norm.
- It is numerically more robust than the FL-BT when reducing an ill-conditioned model.
- As for the FL-BT, the stability on the reduced-order model is no longer guaranteed. However in practice, instability has only be observed for some very specific frequency intervals.
- It is not based on optimality considerations.

3.3 Frequency Limited TSIA

An expression similar to (9) can be written for the $\mathcal{H}_{2,\omega}$-norm, indeed :

$$\begin{aligned}\mathcal{J}_{\mathcal{H}_{2,\omega}} &= \|H(s)\|^2_{\mathcal{H}_{2,\omega}} - \mathbf{trace}\left(2CX_\omega\hat{C}^T\right) + \|\hat{H}(s)\|^2_{\mathcal{H}_{2,\omega}} \\ &= \|H(s)\|^2_{\mathcal{H}_{2,\omega}} + \mathbf{trace}\left(2B^TY_\omega\hat{B}\right) + \|\hat{H}(s)\|^2_{\mathcal{H}_{2,\omega}}\end{aligned} \qquad (15)$$

where X_ω and Y_ω are the solutions of the following Sylvester equations :

$$AX_\omega + X_\omega\hat{A}^T + S(\omega)B\hat{B}^T + B\hat{B}^T\hat{S}^*(\omega) = 0 \qquad (16a)$$

$$A^TY_\omega + Y_\omega\hat{A} - \left(S^*(\omega)C^T\hat{C} + C^T\hat{C}\hat{S}(\omega)\right) = 0 \qquad (16b)$$

where $S(\omega)$ and $\hat{S}(\omega)$ are given by equation (13) with the matrices A and $\hat{A}$, respectively.

Replacing step 4 of Algorithm 2 by the Sylvester equations (16a) and (16b) leads to a frequency-limited version of the TSIA noted FL-TSIA. This algorithm comes from an

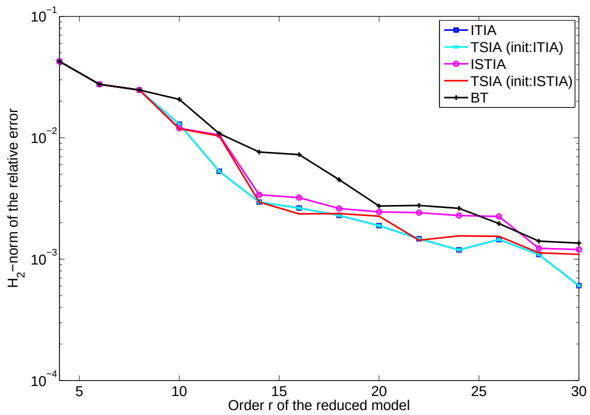

Fig. 1. Reduction of the clamped beam model for several approximation orders r

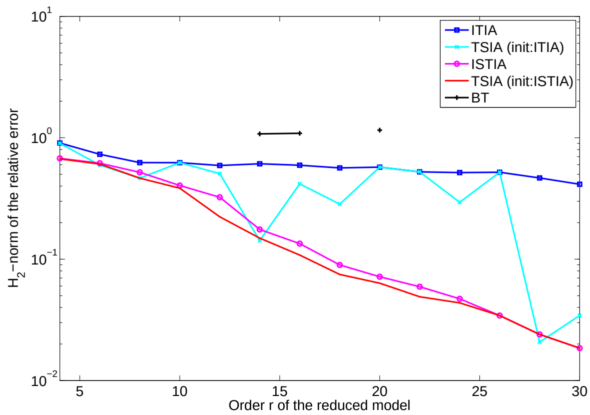

Fig. 2. Reduction of the aircraft model for several approximation orders r

experimental approach and has no theoretical basis yet, but in practice it can substantially decrease the $\mathcal{H}_{2,\omega}$-norm of the error obtained with other methods.

4. APPLICATION

The reduction methods presented above are compared on two models : (i) the clamped beam model, a SISO model with 348 states which can be found in Leibfritz and Lipinski (2003) and (ii) an industrial aircraft model with 289 states, 3 inputs and 4 outputs. The aircraft model is very ill-conditioned and thus quite tricky to reduce. The criterion considered to compare the methods is the relative error, i.e. :

$$\tilde{\mathcal{J}}_{\mathcal{H}_i} = \frac{\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_i}}{\|\Sigma\|_{\mathcal{H}_i}} \tag{17}$$

where $\mathcal{H}_i$ is the $\mathcal{H}_2$-norm in Section 4.1 and the $\mathcal{H}_{2,\omega}$-norm in Section 4.2.

4.1 $\mathcal{H}_2$ model reduction methods

Each model is reduced for different order by the balanced truncation (noted BT and obtained with Matlab®'s reduction tool), the ITIA, the ISTIA and the TSIA (initialized by the ITIA or the ISTIA). Results are presented on Figure 1 for the clamped beam model and on Figure 2 for the aircraft model.

With reference to Figure 1, it can first be noticed that methods which enable to satisfy the first-order optimality conditions (or a subset of them) are better than those which do not, like the BT. However the ITIA is not necessarily better than the ISTIA whereas it satisfies all the optimality conditions. This can be explained by an early convergence towards a local minimum which is related to the choice of stopping criteria. The effect of the TSIA can only be seen when the result of ISTIA is given as initialization, indeed, it does not affect the results of the ITIA.

Differences between the reduction methods appear more clearly on Figure 2. Here the BT leads to a stable reduced-order model in only a few cases (r = 14, 16 or 20). This comes from the ill-conditioning of the matrix A which prevent from finding a good solution to the Lyapunov equations, indeed, $\|A\mathcal{P} + \mathcal{P}A^T + BB^T\|_2 > 10^3$ in some cases. It also seems that the conditioning have a negative impact on the ITIA which gives a bad approximation compared to the ISTIA. This can be explained by the fact that linear systems that have to be solved in the ITIA have not been preconditioned here. Yet it enables to illustrate the benefit of the TSIA which drastically decreases the error of the model obtained through the ITIA for some orders r. The effect is less obvious on the ISTIA but is still there.

4.2 $\mathcal{H}_{2,\omega}$ oriented model reduction methods

Here, each model is reduced on $[0, \omega]$ to a fixed order $r = 12$ by the FL-BT, the ISTIA, the TSIA (initialized by the ISTIA), the FL-ISTIA and the FL-TSIA (initialized by the FL-ISTIA). The reduction is done for several ω : from 2rad/s to 20rad/s for the clamped beam model and from 3rad/s to 40rad/s for the aircraft model. Results are presented on Figures 3 and 4. Note that the FL-BT has not been printed for the aircraft model since it fails to reduce it properly and leads to a large error.

The FL-ISTIA and the FL-TSIA are more efficient in term of $\mathcal{H}_{2,\omega}$-norm than the ISTIA and the TSIA. Indeed, from 2rad/s to 14rad/s for the clamped beam model and from 1rad/s to 19rad/s for the aircraft model, the relative $\mathcal{H}_{2,\omega}$ error is clearly smaller for the frequency-limited approaches. The FL-BT is also better than the ISTIA and the TSIA for the clamped beam model except from 4rad/s to 8rad/s where numerical issues might have appeared. Note also that, similarly to the $\mathcal{H}_2$ case, the FL-TSIA enables to improve the results given by the FL-ISTIA and leads to the best frequency-limited approximation among the tested methods. All the frequency-limited methods quickly converge towards their non frequency-limited versions which was expected since : (i) the frequency-limited gramians converge towards the classical gramians and (ii) the last terms in the Sylvester equations (16a) and (16b) converge towards the last terms of the Sylvester equations in step 4 of Algorithm 2.

5. CONCLUSION

In this paper, new approaches to achieve numerically robust and stable frequency-limited model order reduc-

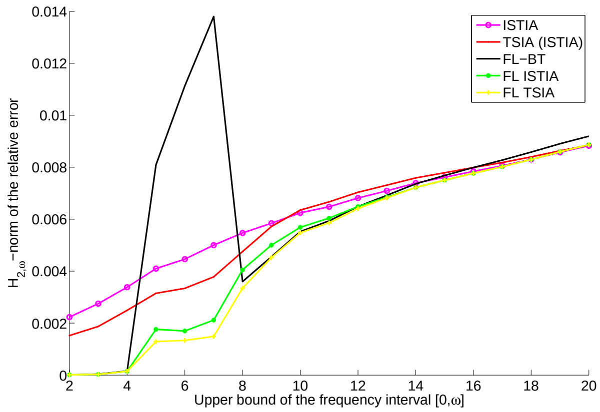

Fig. 3. Frequency-limited reduction of the clamped beam model (r = 12) for varying ω

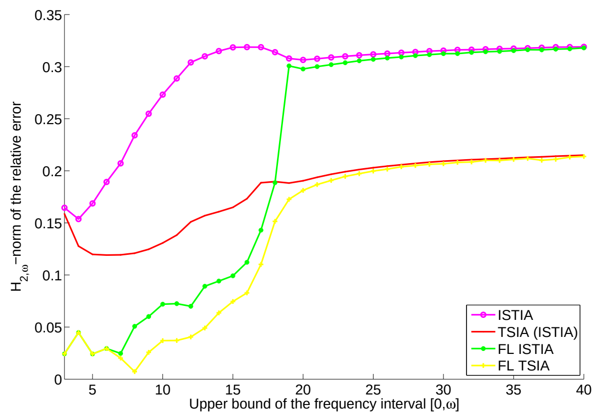

Fig. 4. Frequency-limited reduction of the aircraft model (r = 12) for varying ω

tion through weight-free algorithms have been proposed, namely the Frequency-Limited ISTIA and the Frequency-Limited TSIA. The theoretical background is still under investigation, but in practice, it is clear that these methods lead to good approximation in terms of $\mathcal{H}_{2,\omega}$-norm. In fact, the approximations are at least as good as those obtained with the FL-BT, the other well known weight-free method. The FL-ISTIA requires to solve a Lyapunov equation and is thus dedicated to medium-scale models whereas the FL-TSIA, provided it has been well initialised, can be successfully applied to larger models. The FL-ISTIA and the FL-TSIA will be available soon in the MORE Toolbox (see Poussot-Vassal and Vuillemin (2012)).

REFERENCES

Anic, B., Beattie, C., Gugercin, S., and Antoulas, A.C. (2012). Interpolatory Weighted-$\mathcal{H}_2$ Model Reduction. Submitted to Automatica.

Antoulas, A.C. (2005). *Approximation of Large-Scale Dynamical Systems*. Society for Industrial and Applied Mathematics.

Beattie, C. and Gugercin, S. (2011). Weighted model reduction via interpolation. In *Proceedings of the 18th IFAC World Congress*, 12757–12760. Milano, Italy.

Benner, P., Khler, M., and Saak, J. (2011). Sparse-dense sylvester equations in $\mathcal{H}_2$ model order reduction. Technical report, Max Planck Institute Magdeburg.

Gallivan, K.A., Vandendorpe, A., and Van Dooren, P. (2004). Model reduction of MIMO systems via tangential interpolation. *SIAM J. Matrix Anal. Appl.*, 26(2), 328–349.

Gawronski, W. and Juang, J. (1990). Model reduction in limited time and frequency intervals. *International Journal of Systems Science*, 21(2), 349–376.

Grimme, E.J. (1997). *Krylov Projection Methods For Model Reduction*. Ph.D. thesis, University of Illinois at Ubarna-Champaign.

Gugercin, S. (2007). An iterative SVD-Krylov based method for model reduction of large-scale dynamical systems. *Linear Algebra and its Applications*, 428(89), 1964–1986.

Gugercin, S. and Antoulas, A.C. (2004). A survey of model reduction by balanced truncation and some new results. *International Journal of Control*, 77(8), 748–766.

Gugercin, S., Antoulas, A.C., and Beattie, C. (2008). $\mathcal{H}_2$ model reduction for Large-Scale linear dynamical systems. *SIAM Journal on Matrix Analysis and Applications*, 30(2), 609–638.

Leibfritz, F. and Lipinski, W. (2003). Description of the benchmark examples in *COMPl$_e$ib* 1.0. Technical report, University of Trier.

Masi, A., Wallin, R., Garulli, A., and Hansson, A. (2010). Robust finite-frequency $\mathcal{H}_2$ analysis. In *Proceedings of the 49th IEEE Conferance on Decision and Control*, 6876 –6881.

Meier, L. and Luenberger, D. (1967). Approximation of linear constant systems. *IEEE Transactions on Automatic Control*, 12(5), 585–588.

Petersson, D. and Löfberg, J. (2012). Model reduction using a frequency-limited $\mathcal{H}_2$-cost. Technical report, Linkpings universitet.

Poussot-Vassal, C. (2011). An Iterative SVD-Tangential Interpolation Method for Medium-Scale MIMO Systems Approximation with Application on Flexible Aircraft. In *Proceedings* 50th *IEEE Conference on Decision and Control - European Control Conference (CDC-ECC'11)*, 7117–7122. Florida.

Poussot-Vassal, C. and Vuillemin, P. (2012). Introduction to MORE: a MOdel REduction Toolbox. To appear in Proceedings of the IEEE Multi Systems Conference (MSC CCA'12), Dubrovnik, Croatia.

Van Dooren, P., Gallivan, K.A., and Absil, P.A. (2008). $\mathcal{H}_2$-optimal model reduction of MIMO systems. *Applied Mathematics Letters*, 21(12), 1267–1273.

Vuillemin, P., Poussot-Vassal, C., and Alazard, D. (2012). A Frequency-Limited Iterative SVD-Tangential Medium-Scale Model Approximation Method. Submitted to the 51st IEEE Conference on Decision and Control.

Wilson, D.A. (1974). Model reduction for multivariable systems. *International Journal of Control*, 20(1), 57–64.

Xu, Y. and Zeng, T. (2011). *International Journal of Numerical Analysis and Modeling*, 8(1), 174–188.