Spectral expression for the Frequency-Limited $\mathcal{H}_2$-norm of LTI Dynamical Systems with High Order Poles

Pierre Vuillemin, Charles Poussot-Vassal and Daniel Alazard

Abstract— **In this paper, the spectral expression of the frequency-limited $\mathcal{H}_2$-norm, firstly presented in [18], is extended to the case of LTI dynamical systems which are not diagonalizable. This extension is achieved by considering partial fraction decomposition of the transfer function associated to a system which is in a Jordan form. Besides, examples are presented to illustrate the behaviour of the frequency-limited $\mathcal{H}_2$-norm and to compare it with the commonly used frequency-weighted $\mathcal{H}_2$-norm.**

I. Introduction

A. Context & contributions

Norms associated to Multiple Inputs Multiple Outputs (MIMO) LTI dynamical systems, such as the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms, are of great interest in system theory. They are often used as cost functions in controller and observer design [16], [1], [5], [7] or in large-scale model approximation [8], [11], [6]. This paper is concerned with the restriction of the $\mathcal{H}_2$-norm over a bounded frequency range, namely the frequency-limited $\mathcal{H}_2$-norm, denoted here $\mathcal{H}_{2,\Omega}$-norm, and more specifically to its computation though spectral information. This measure has been mentionned in [2] and is related to the frequency-limited gramians proposed in [9]. In practice, the $\mathcal{H}_{2,\Omega}$-norm is of interest when the whole frequency behaviour of a system is not accurately known or not needed. Indeed *(i)* the accuracy of the model representing a physical system is dependent of the sensors bandwidth which is not always well known over high frequencies and *(ii)* due to the limited actuators bandwidth, controllers can only act in a limited frequency range.

Throughout this paper, a stable and strictly proper MIMO LTI dynamical system $\mathbf{H}$ is considered. It is defined as

$$\mathbf{H} := \left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) &=& A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &=& C\mathbf{x}(t) \end{array} \right. \quad (1)$$

where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times n_u}$ and $C \in \mathbb{R}^{n_y\times N}$. The associated transfer function $H(s)$ of $\mathbf{H}$ is given by,

$$H(s) = C(sI_N - A)^{-1}B \in \mathbb{C}^{n_y\times n_u}. \quad (2)$$

The $\mathcal{H}_{2,\Omega}$-norm of $\mathbf{H}$, denoted $\|H\|_{\mathcal{H}_{2,\Omega}}$ is given in Definition 1.

Pierre Vuillemin and Daniel Alazard are with Université de Toulouse and Onera - The French Aerospace Lab, F-31055 Toulouse, France, pierre.vuillemin@onera.fr.

Charles Poussot-Vassal is with Onera - The French Aerospace Lab, F-31055 Toulouse, France.

Definition 1 ($\mathcal{H}_{2,\Omega}$-norm): The $\mathcal{H}_{2,\Omega}$-norm of a LTI dynamical system $\mathbf{H}$, is defined as the restriction of the $\mathcal{H}_2$-norm over $\Omega = [0, \omega]$, $\omega \in \mathbb{R}_+^*$, *i.e.*

$$\|H\|_{\mathcal{H}_{2,\Omega}} = \sqrt{\frac{1}{2\pi}\int_{-\omega}^{\omega} \mathbf{tr}\left(H(\mathbf{j}\nu)H(-\mathbf{j}\nu)^T\right) d\nu}. \quad (3)$$

Note that a much more complex frequency interval Ω can easily be considered in Definition 1, for instance $\Omega = \bigcup_{k=1}^{K}\left[\omega_1^{(k)}\ \omega_2^{(k)}\right]$, $\omega_1^{(k)} < \omega_2^{(k)} < \omega_1^{(k+1)}$ where $\omega_2^{(K)}$ can be infinite if the system is strictly proper. As a matter of fact, if $\Omega = [\omega_1, \omega_2]$,

$$\|H\|_{\mathcal{H}_{2,\Omega}}^2 = \|H\|_{\mathcal{H}_{2,\Omega_2}}^2 - \|H\|_{\mathcal{H}_{2,\Omega_1}}^2, \quad (4)$$

where $\Omega_1 = [0, \omega_1]$ and $\Omega_2 = [0, \omega_2]$. Here, for sake of simplicity, only $\Omega = [0, \omega]$ is considered.

Similarly to the $\mathcal{H}_2$-norm, the $\mathcal{H}_{2,\Omega}$-norm can be expressed with the system's frequency-limited gramians [9]. Indeed, let consider the frequency-limited controllability and observability gramians $\mathcal{P}_\Omega$ and $\mathcal{Q}_\Omega$, respectively, of the LTI dynamical system $\mathbf{H}$. They are defined as the restriction of the infinite gramians over Ω, *i.e.*

$$\begin{array}{rcl} \mathcal{P}_\Omega &=& \dfrac{1}{2\pi}\displaystyle\int_{-\omega}^{\omega} T(\nu)BB^T T^*(\nu) d\nu \\ \mathcal{Q}_\Omega &=& \dfrac{1}{2\pi}\displaystyle\int_{-\omega}^{\omega} T^*(\nu)C^T C T(\nu) d\nu \end{array} \quad (5)$$

where $T(\nu) = (\mathbf{j}\nu I_n - A)^{-1}$. Then the frequency-limited $\mathcal{H}_2$-norm of $\mathbf{H}$ is straightforwardly given by

$$\|H\|_{\mathcal{H}_{2,\Omega}}^2 = \mathbf{tr}\left(C\mathcal{P}_\Omega C^T\right) = \mathbf{tr}\left(B^T\mathcal{Q}_\Omega B\right). \quad (6)$$

Under some additional assumptions, the $\mathcal{H}_{2,\Omega}$-norm can also be expressed with the transfer function's poles and residues. In this paper, the latter formulation is considered for systems which have Jordan blocks larger than one which extends the result presented in [18] by alleviating some assumptions.

B. Motivating examples

The frequency-limited $\mathcal{H}_2$-norm has mainly been used in analysis and large-scale model approximation, *e.g.* :

- In [2], this metric is suggested to get information on the frequency response of nominally unstable systems.
- In [14], the $\mathcal{H}_{2,\Omega}$-norm has been used to perform robustness analysis.
- In [16], it has been used to perform comfort analysis of an industrial aircraft aeroelastic model.
- In [9], the frequency-limited gramians are used to perform a frequency-limited balanced truncation.

- Optimal frequency-limited model approximation is performed in [15] with the gramian formulation of the $\mathcal{H}_{2,\Omega}$-norm and in [19] with the poles-residues formulation of the norm for models with a diagonalizable matrix A.
- In [20], two upper bounds on the $\mathcal{H}_\infty$-norm of LTI dynamical systems are derived from the $\mathcal{H}_{2,\Omega}$-norm.

Note that in the large-scale model reduction framework, the usual approach for the reduced-order model to accurately reproduce the behaviour of the initial model over a bounded frequency range consists in applying weighting filters, leading to the so called *frequency-weighted* model reduction problem (see for instance [10] and references therein). Recently, this yields to first-order optimality conditions with respect to the weighted $\mathcal{H}_2$-norm [3]. Yet the $\mathcal{H}_{2,\Omega}$-norm has two main advantage over the frequency-weighted $\mathcal{H}_2$-norm :

- The $\mathcal{H}_{2,\Omega}$-norm is equivalent to the frequency-weighted $\mathcal{H}_2$-norm computed with perfect filters. Hence the $\mathcal{H}_{2,\Omega}$-norm is more accurate and does not require to design filters. This difference can be observed in [10] where the frequency-balanced truncation is proven to be equivalent to the frequency-weighted balanced truncation done with perfect filters.
- The $\mathcal{H}_{2,\Omega}$-norm can be computed for systems with a direct feedthrough D which is not the case of the frequency-weighted $\mathcal{H}_2$-norm [18].

Moreover, the interest of the poles/residues formulation of the norm also comes from model approximation. Indeed, in the $\mathcal{H}_2$ model approximation problem, the poles/residues expression of the $\mathcal{H}_2$-norm [4, chap.5] has enabled to express first-order optimality conditions as convenient interpolation conditions between transfer functions and has led to efficient iterative algorithms [11], [17].

C. Notations & Paper structure

The notations used throughout this paper are the following : $\mathbf{tr}\left(A\right)$ represents the trace of the matrix A, A^T is the transpose of A, $A \odot B$ is the Hadamard product between A and B, the bold $\mathbf{j}$ denotes the complex variable, $ln(x)$ is the natural logarithm of $x \in \mathbb{R}_+^*$, $\mathbf{log}(z)$ denotes the principal value of the complex logarithm of $z \neq \pm 0$, $\mathbf{atan}(z)$ is the complex inverse tangent function of $z \neq \pm \mathbf{j}$, λ_i and ϕ_i denotes the eigenvalues and associated residues of a system $\mathbf{H}$, respectively and $\omega > 0$ is a pulsation in rad/sec.

This paper is divided as follows : in Section II, preliminary results concerning the $\mathcal{H}_{2,\Omega}$-norm of systems with simple poles are recalled and an illustration of the $\mathcal{H}_{2,\Omega}$-norm as well as a comparison with the frequency-weighted $\mathcal{H}_2$-norm are presented. Then, in Section III, the poles/residues formulation of the $\mathcal{H}_{2,\Omega}$-norm is extended to systems which have Jordan blocks larger than one. Finally, Section IV concludes this article and draw potential perspectives.

II. Preliminary results

A. Spectral expression of the $\mathcal{H}_{2,\Omega}$-norm : the diagonalizable case

It is well known that if the matrix A of (1) is diagonalizable, then the partial fraction decomposition of the transfer function $H(s)$ is

$$H(s) = \sum_{i=1}^{N} \frac{\phi_i}{s - \lambda_i}, \tag{7}$$

where $\lambda_i \in \mathbb{C}$, $\phi_i \in \mathbb{C}^{n_y \times n_u}$ $(i = 1, \ldots, N)$ are the poles and associated residues of $H(s)$. The residues $\phi_i \in \mathbb{C}^{n_y \times n_u}$ of $H(s)$ are defined, for $i = 1, \ldots, N$, as

$$\phi_i = \lim_{s \to \lambda_i} (s - \lambda_i)\, H(s), \tag{8}$$

and can be obtained by computing the right eigenvectors $X = \begin{bmatrix} \mathbf{x_1} & \cdots & \mathbf{x_N} \end{bmatrix} \in \mathbb{C}^{N \times N}$ of A. Indeed, by denoting $\mathbf{c_i}^T \in \mathbb{C}^{n_y \times 1}$ the i-th column of CX and $\mathbf{b_i} \in \mathbb{C}^{1 \times n_u}$ the i-th line of $X^{-1}B$, it turns out that, for $i = 1, \ldots, N$,

$$\phi_i = \mathbf{c_i}^T \mathbf{b_i}. \tag{9}$$

The decomposition (7) is the basis of the result presented in [18] which simplest form is recalled in Theorem 1.

Theorem 1: Given a N-th order stable and strictly proper MIMO LTI dynamical system $\mathbf{H} := (A, B, C)$ which transfer function is $H(s)$ and an interval $\Omega = [0, \omega]$ with $\omega > 0$. If A is diagonalizable, then the frequency-limited $\mathcal{H}_2$-norm of $\mathbf{H}$, denoted $\|H\|_{\mathcal{H}_{2,\Omega}}$, is given by

$$\|H\|^2_{\mathcal{H}_{2,\Omega}} = \sum_{i=1}^{N} \mathbf{tr}\left(\phi_i H(-\lambda_i)\right) \left(-\frac{2}{\pi}\mathbf{atan}\left(\frac{\omega}{\lambda_i}\right)\right), \tag{10}$$

where λ_i, ϕ_i, $i = 1, \ldots, N$ are the poles and associated residues of $H(s)$, respectively.

Remark 1 (The complex arctangent): The complex arctangent function appearing in (10) is defined, for $z \neq \pm \mathbf{j}$, as

$$\mathbf{atan}\left(z\right) = \frac{1}{2\mathbf{j}}\left(\mathbf{log}\left(1 + \mathbf{j}z\right) - \mathbf{log}\left(1 - \mathbf{j}z\right)\right), \tag{11}$$

where $\mathbf{log}(z)$ is the natural logarithm of z defined, for $z \neq 0$, as,

$$\mathbf{log}\left(z\right) = ln(|z|) + \mathbf{j}\,\mathbf{arg}(z) \tag{12}$$

with $-\pi < \mathbf{arg}(z) \leq \pi$ and $ln(x)$ the natural logarithm of $x \in \mathbb{R}_+^*$. There exists another definition of the complex arctangent, but since the system is assumed to be stable, both definitions are equivalent (see [12] for further information).

Since $H(-\lambda_i)$ in equation (10) can be replaced by

$$H(-\lambda_i) = \sum_{k=1}^{N} \frac{\phi_k}{-\lambda_i - \lambda_k}, \tag{13}$$

the $\mathcal{H}_{2,\Omega}$-norm of $\mathbf{H}$ can be rewritten as

$$\|H\|^2_{\mathcal{H}_{2,\Omega}} = \sum_{i=1}^{N}\sum_{k=1}^{N} \frac{\mathbf{tr}\left(\phi_i \phi_k^T\right)}{\lambda_i + \lambda_k}\mathbf{atan}\left(\frac{\omega}{\lambda_i}\right). \tag{14}$$

By denoting $X \in \mathbb{C}^{N \times N}$ the matrix which columns are the right eigenvectors of A, $Y = X^{-1}$ and $e_i \in \mathbb{R}^{N \times 1}$ the canonical basis vector, it turns out that

$$\mathbf{tr}\left(\phi_i \phi_k^T\right) = e_k^T \left(CX\right)^T CXe_i e_i^T YB\left(YB\right)^T e_k. \tag{15}$$

Due to the symmetry arising in equation (15), the $\mathcal{H}_{2,\Omega}$-norm of $\mathbf{H}$ can be efficiently computed through the following expression :

$$\|H\|_{\mathcal{H}_{2,\Omega}}^2 = \mathbb{1}^T \left(M_1 \odot M_2 \odot L\right) \begin{bmatrix} -\frac{2}{\pi}\mathbf{atan}\left(\frac{\omega}{\lambda_1}\right) \\ \vdots \\ -\frac{2}{\pi}\mathbf{atan}\left(\frac{\omega}{\lambda_N}\right) \end{bmatrix}, \tag{16}$$

where

$$M_1 = (CX)^T CX \quad \text{and} \quad M_2 = YB(YB)^T, \tag{17}$$

and for $i, k = 1, \ldots, N$,

$$[L]_{i,k} = \frac{1}{\lambda_i + \lambda_k}. \tag{18}$$

B. Relation with the $\mathcal{H}_2$-norm

For stable and strictly proper systems, the $\mathcal{H}_{2,\Omega}$-norm is related to the $\mathcal{H}_2$-norm as presented in Property 1.

Property 1: Let consider a stable and strictly proper MIMO LTI dynamical system $\mathbf{H}$ and an interval $\Omega = [0, \omega]$, $\omega > 0$, then the $\mathcal{H}_{2,\Omega}$-norm of $\mathbf{H}$ tends towards its $\mathcal{H}_2$ norm as ω tends towards infinity, *i.e.*

$$\lim_{\omega \to \infty} \|H\|_{\mathcal{H}_{2,\Omega}} = \|H\|_{\mathcal{H}_2}. \tag{19}$$

For a stable and strictly proper system, by considering the limit of the complex inverse tangent function [12] in the poles-residues expression of the $\mathcal{H}_{2,\Omega}$-norm (10) as ω tends towards infinity, *i.e.*

$$\lim_{\omega \to \infty} \mathbf{atan}\left(\frac{\omega}{\lambda}\right) = -\frac{\pi}{2} \Leftrightarrow \mathbf{Re}\left(\lambda\right) < 0, \tag{20}$$

the poles-residues expression of the $\mathcal{H}_2$-norm presented in [4, chap. 5],

$$\|H\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} \mathbf{tr}\left(\phi H(-\lambda_i)^T\right), \tag{21}$$

is recovered. The complex inverse tangent functions arising in (10) thus play the role of optimal filters on each pole contribution.

C. Illustration of the $\mathcal{H}_{2,\Omega}$-norm and comparison with the frequency-weighted $\mathcal{H}_2$-norm

To illustrate the behaviour of the frequency-limited $\mathcal{H}_2$ norm, the Los-Angeles Hospital (LAH) model available in [13] is used. It is a 48-th order stable and strictly proper SISO dynamical system with simple poles only.

The $\mathcal{H}_{2,\Omega}$-norm of the LAH model is computed for $\Omega = [0, \omega]$ with $\omega \in [1, 100]$ and plotted in Figure 1 together with the gain of the frequency response.

The following remarks can be made :

- As expected, the $\mathcal{H}_{2,\Omega}$-norm tends towards the $\mathcal{H}_2$-norm as ω increases.

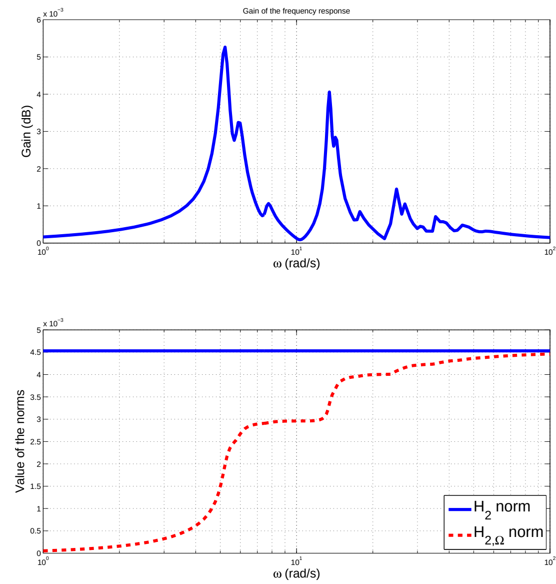

Fig. 1: Gain of the frequency response of the LAH model (top) and $\mathcal{H}_{2,\Omega}$-norm of the model for $\Omega = [0, \omega]$ with $\omega \in [1, 100]$ (bottom).

- The evolution of the $\mathcal{H}_{2,\Omega}$-norm gives insight about the frequency behaviour of the system. Indeed, the $\mathcal{H}_{2,\Omega}$-norm increases quickly when ω crosses frequencies corresponding to high gains. This property is used in [20] to derive upper bounds to the $\mathcal{H}_\infty$-norm of LTI dynamical systems.

In the following example, the frequency-limited $\mathcal{H}_2$-norm of the LAH model is computed over the frequency interval $\Omega = [10, 20]$ and compared to the $\mathcal{H}_2$-norm computed on the weighted system obtained by applying an input bandpass filter to the system. The filter is constructed with two butterworth filters which orders are increased from 0 to 10. The top frame of Figure 2 shows the $\mathcal{H}_{2,\Omega}$-norm and frequency-weighted $\mathcal{H}_2$-norm for varying order of the bandpass filter and the bottom frame represents the relative error of the frequency-weighted $\mathcal{H}_2$-norm compared to the $\mathcal{H}_{2,\Omega}$-norm.

The frequency-weighted $\mathcal{H}_2$-norm tends towards the $\mathcal{H}_{2,\Omega}$-norm as the order of the filter increases. With a 8-th order bandpass filter, the relative error falls below 5%. The frequency-weighted $\mathcal{H}_2$-norm is not necessarily inferior or superior to the $\mathcal{H}_{2,\Omega}$-norm, both cases can be observed, depending on the system. Note that the required order of the filter strongly depends on the considered system and the frequency interval Ω. Besides, multiple frequency intervals might be difficult to handle with filters whereas they are indifferently handled with the $\mathcal{H}_{2,\Omega}$-norm.

III. Extension to Higher order poles

Theorem 1 relies on the formulation (7) of the transfer function which exists only for systems with a diagonalizable

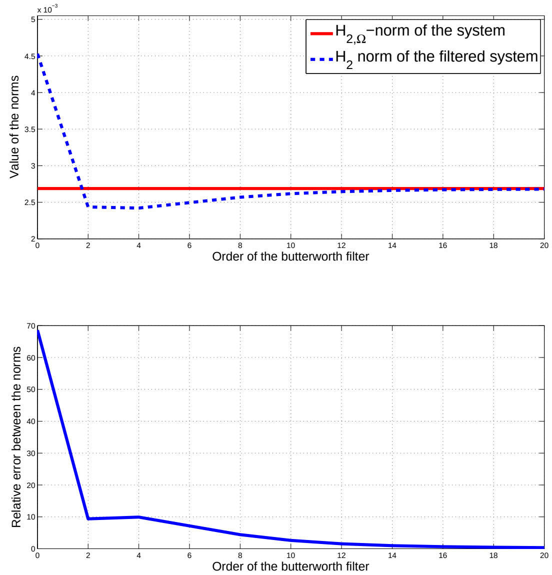

Fig. 2: Comparison of the frequency-weighted $\mathcal{H}_2$-norm and the frequency limited $\mathcal{H}_2$-norm of the LAH model over $\Omega = [\omega_1, \omega_2]$.

matrix A. If A is not diagonalizable, then another formulation must be used, as presented in this section.

A. Decomposition of the transfer function

Let us now consider the more general case of a system $\mathbf{H}$ with n_b stable poles λ_i of multiplicity n_i such that $\sum_{i=1}^{n_b} n_i = N$. In this case, the partial fraction decomposition of $H(s)$ is given by

$$H(s) = \sum_{i=1}^{n_b} H_{\lambda_i}(s) = \sum_{i=1}^{n_b}\sum_{j=1}^{n_i} \frac{\phi_{ij}}{(s-\lambda_i)^j}, \qquad (22)$$

where the $\phi_{ij} \in \mathbb{C}^{n_y \times n_u}$, $j = 1, \ldots, n_i$ are the residues associated with the pole λ_i, *i.e.*

$$\phi_{ij} = \lim_{s\to\lambda_i} \frac{1}{(n_i-j)!}\frac{d^{n_i-j}}{ds^{n_i-j}}(s-\lambda_i)^{n_i} H(s). \qquad (23)$$

Again, these residues are linked to the state-space representation of $\mathbf{H}$. Indeed, let consider the transformation $T \in \mathbb{C}^{N\times N}$ which transforms the matrix A in a Jordan canonical form, *i.e.*

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_{n_b} \end{bmatrix}, \qquad (24)$$

where for $i = 1, \ldots, n_b$,

$$T_i^{-1}AT_i = J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i\times n_i}, \qquad (25)$$

is the i-th Jordan block of size n_i associated with the eigenvalue λ_i. Then the associated transfer functions $H_{\lambda_i}(s)$ are given, for $i = 1, \ldots, n_b$, by

$$H_{\lambda_i}(s) = CT_i(sI_{n_i} - J_i)^{-1}T_i^{-1}B. \qquad (26)$$

The structure of the matrices J_i enables to write the inverse in (26) as a sum of rational functions of s,

$$\begin{aligned}(sI_{n_i} - J_i)^{-1} &= (s-\lambda_i)^{-1}F_1 + (s-\lambda_i)^{-2}F_2 + \ldots \\ & \quad \ldots + (s-\lambda_i)^{-n_i}F_{n_i}\end{aligned} \qquad (27)$$

where $F_j \in \mathbb{R}^{n_i\times n_i}$ is the matrix with 1 on the $(j-1)$-th superior diagonal and 0 elsewhere, *i.e.*

$$F_1 = I_{n_i},\; F_2 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \ldots, F_{n_i} = \begin{bmatrix} 0 & 0 & & 1 \\ & 0 & \ddots & \\ & & \ddots & 0 \\ & & & 0 \end{bmatrix} \qquad (28)$$

or more briefly, by denoting $\mathbf{e_k} \in \mathbb{R}^{n_i\times 1}$ the k-th canonical basis vector,

$$F_j = \sum_{k=1}^{n_i-j+1} \mathbf{e_k}\mathbf{e_{k+j-1}}^T. \qquad (29)$$

Applying (27) and (29) in (26), it comes that for $i = 1, \ldots, n_b$,

$$H_{\lambda_i}(s) = \sum_{j=1}^{n_i} \frac{1}{(s-\lambda_i)^j} \underbrace{\sum_{k=d(i)+1}^{d(i)+n_i-j+1} \mathbf{c_k}^T\mathbf{b_{k+j-1}}}_{\phi_{ij}}, \qquad (30)$$

where $\mathbf{c_k}^T \in \mathbb{C}^{n_y\times 1}$ is the k-th column of CT, $\mathbf{b_k} \in \mathbb{C}^{1\times n_u}$ the k-th line of $T^{-1}B$ and $d(i)$ an index shift defined as

$$d(i) = \begin{cases} 0 & \text{if } i = 1 \\ \sum_{l=1}^{i-1} n_i & \text{otherwise.} \end{cases} \qquad (31)$$

The index shift $d(i)$ is necessary to select the right vectors $\mathbf{c_k}$ and $\mathbf{b_k}$. For instance, for $i = 1$, k varies between 1 and n_1 which correspond to the first Jordan bock, whereas for $i = 2$, the first n_1 vectors must not be used and k must vary between $n_1 + 1$ and $n_1 + n_2$. Finally, when $i = n_b$, the residues corresponding to the last Jordan block are considered, thus k varies between $n_1 + n_2 + \ldots + n_{n_b-1} + 1$ and $n_1 + n_2 + \ldots + n_{n_b}$

B. Spectral expression of the $\mathcal{H}_{2,\Omega}$-norm : the general case

Based on the formulation (26) of the system's transfer function, Theorem 1 is generalized to higher order poles in Theorem 2.

Theorem 2: Given a N-th order stable and strictly proper MIMO LTI dynamical system $\mathbf{H} := (A, B, C)$ which transfer function is $H(s)$ and an interval $\Omega = [0, \omega]$ with $\omega > 0$. Let $\mathbf{H}$ have n_b eigenvalues λ_i of multiplicity n_i, then the frequency-limited $\mathcal{H}_2$-norm of H is given by

$$\|H\|^2_{\mathcal{H}_{2,\Omega}} = \frac{\mathbf{j}}{2\pi}\sum_{i,j=1}^{n_b}\sum_{k=1}^{n_i}\sum_{l=1}^{n_j} \mathbf{tr}\left(\phi_{ik}\phi_{jl}^T\right) I_{ijkl}(\omega) \qquad (32)$$

with

$$I_{ijkl}(\omega) = \begin{array}{l} \sum_{m=1}^{k} r_{ij}(l,k-m) W_{m-1}\left(j\omega,\lambda_i\right)\dots \\ +\sum_{n=1}^{l} r_{ij}(k,l-n) W_{n-1}\left(j\omega,\lambda_j\right), \end{array} \tag{33}$$

where

$$W_p(z,\lambda) = \frac{1}{p!}\frac{\partial^p}{\partial y^p}\left(\mathbf{log}\left(-x-y\right)-\mathbf{log}\left(x-y\right)\right)\Big|_{\substack{x=z\\y=\lambda}}, \tag{34}$$

and

$$r_{ij}(p,q) = (-1)^{(p+q)}\begin{pmatrix} p+q-1 \\ q \end{pmatrix}\frac{1}{\left(\lambda_i+\lambda_j\right)^{p+q}}. \tag{35}$$

Proof: Let consider a stable and strictly proper MIMO LTI dynamical system $\mathbf{H}$ with n_b eigenvalues λ_i of multiplicity n_i described by its transfer function $H(s)$. The $\mathcal{H}_{2,\Omega}$-norm of $\mathbf{H}$ is defined as,

$$\|H\|^2_{\mathcal{H}_{2,\Omega}} = \frac{1}{2\mathbf{j}\pi}\int_{-\mathbf{j}\omega}^{\mathbf{j}\omega}\mathbf{tr}\left(H(s)H(-s)^T\right)ds. \tag{36}$$

By replacing $H(s)$ by its partial fraction expansion (22), it comes that

$$\mathbf{tr}\left(H(s)H(-s)^T\right) = \sum_{i,j=1}^{n_b}\sum_{k=1}^{n_i}\sum_{l=1}^{n_j}\frac{\mathbf{tr}\left(\phi_{ik}\phi_{jl}^T\right)}{\left(s-\lambda_i\right)^k\left(-s-\lambda_j\right)^l}. \tag{37}$$

The integral term of (36) comes down to the following integral for each i, j, k and l,

$$\int_{-\mathbf{j}\omega}^{\mathbf{j}\omega}\frac{\mathbf{tr}\left(\phi_{ik}\phi_{jl}^T\right)}{\left(s-\lambda_i\right)^k\left(-s-\lambda_j\right)^l}ds = \mathbf{tr}\left(\phi_{ik}\phi_{jl}^T\right)\underbrace{\int_{-\mathbf{j}\omega}^{\mathbf{j}\omega}f_{ijkl}(s)ds}_{I_{ijkl}(\omega)}. \tag{38}$$

By noticing that the functions $f_{ijkl}(s)$ has 2 poles, λ_i and $-\lambda_j$, of order k and l, respectively, their partial fraction decomposition are given by

$$f_{ijkl}(s) = \sum_{m=1}^{k}\frac{a_m}{\left(s-\lambda_i\right)^m} + \sum_{n=1}^{l}\frac{b_n}{\left(-s-\lambda_j\right)^n}, \tag{39}$$

where

$$a_m = \frac{1}{(k-m)!}\frac{d^{k-m}}{ds^{k-m}}\left(s-\lambda_i\right)^k f_{ijkl}(s)\Big|_{s=\lambda_i} \tag{40}$$

for $m = 1,\dots,k$ and

$$b_n = (-1)^{l-n}\frac{1}{(l-n)!}\frac{d^{l-n}}{ds^{l-n}}\left(-s-\lambda_j\right)^l f_{ijkl}(s)\Big|_{s=-\lambda_j}, \tag{41}$$

for $l = 1,\dots,n$. Note that the sign $(-1)^{l-n}$ is introduced due to the specific form of the partial fraction decomposition (39) which uses $\frac{1}{(-s-\lambda_j)^n}$ instead of $(-1)^l\frac{1}{(s+\lambda_j)^n}$. The residues a_m and b_n can be written in similar forms. Indeed $a_m = r_{ij}(l,k-m)$ and $b_n = r_{ij}(k,l-n)$ where $r_{ij}(p,q)$ is given in (35).

Since the system $\mathbf{H}$ is stable, each integral composing $I_{ijkl}(\omega)$ can be directly calculated. Indeed,

$$\begin{aligned} \int_{-\mathbf{j}\omega}^{\mathbf{j}\omega}\frac{a_1}{s-\lambda_i} &= a_1\left[\mathbf{log}\left(s-\lambda_i\right)\right]_{-\mathbf{j}\omega}^{\mathbf{j}\omega}, \\ \int_{-\mathbf{j}\omega}^{\mathbf{j}\omega}\frac{a_2}{\left(s-\lambda_i\right)^2} &= a_2\left[-\frac{1}{s-\lambda_i}\right]_{-\mathbf{j}\omega}^{\mathbf{j}\omega}, \end{aligned} \tag{42}$$

and so on for each value of $m = 1,\dots,k$ and $n = 1,\dots,l$. The resulting functions of ω can be written in a more convenient way as $W_{m-1}(\mathbf{j}\omega,\lambda_i)$ and $W_{n-1}(\mathbf{j}\omega,\lambda_j)$ where $W_p(z,\lambda)$ is defined in (34). ■

C. Property

Given a stable and strictly proper system $\mathbf{H}$ with one simple Jordan block of size n associated to the eigenvalue λ which corresponding residues are ϕ_l, $l = 1,\dots,n$, the expression presented in [4, chap. 5],

$$\|H\|^2_{\mathcal{H}_2} = \mathbf{tr}\left(\sum_{i=1}^{n}\frac{\phi_i}{(i-1)!}\frac{d^{i-1}}{ds^{i-1}}H(-s)^T\Big|_{s=\lambda}\right), \tag{43}$$

is retrieved when ω tends towards infinity from equation (32). Indeed, it is straightforward to notice that, for $p > 0$,

$$\lim_{\omega\to\infty}|W_p(\mathbf{j}\omega,\lambda)| = 0, \tag{44}$$

hence, by noting that

$$\lim_{\omega\to\infty}W_0(j\omega,\lambda) = -\mathbf{j}\pi, \tag{45}$$

it comes that

$$\lim_{\omega\to\infty}I_{ijkl}(\omega) = -2\mathbf{j}\pi r_{ij}(l,k-1). \tag{46}$$

When one single eigenvalue of order n is considered, $n_b = 1$ and $n_i = n_j = n$, thus

$$\lim_{\omega\to\infty}\|H\|^2_{\mathcal{H}_{2,\Omega}} = \sum_{k=1}^{n}\sum_{l=1}^{n}\mathbf{tr}\left(\phi_k\phi_l^T\right)r(l,k-1), \tag{47}$$

where $r(l,k-1) = (-1)^{l+k-1}\begin{pmatrix} l+k-2 \\ k-1 \end{pmatrix}\frac{1}{(2\lambda)^{l+k-1}}$. Obviously,

$$\sum_{l=1}^{n}\phi_l^T r(l,k-1) = \frac{1}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}H(-s)^T\Big|_{s=\lambda}, \tag{48}$$

which leads to the expression (43). This short property illustrates the fact that the proposed approach extends the previous formulation.

D. Illustrative example

In this example, a Jordan form is constructed by choosing 4 arbitrary eigenvalues $\lambda_1 = -2$, $\lambda_2 = -0.3 \pm \mathbf{j}$, $\lambda_3 = -0.3 \pm 5\mathbf{j}$ and $\lambda_4 = -0.4 \pm 10\mathbf{j}$ of order 3 (2 Jordan blocks of size 1 and 2), 2 (1 block of size 2), 5 (2 blocks of size 3 and 2) and 6 (3 blocks of size 2), respectively. The resulting model is thus a 29-th order one. The B and C matrices are chosen as vectors full of ones (the MIMO case is handled indifferently). The generalized eigenvectors forming the matrix T (24) are chosen randomly but they must

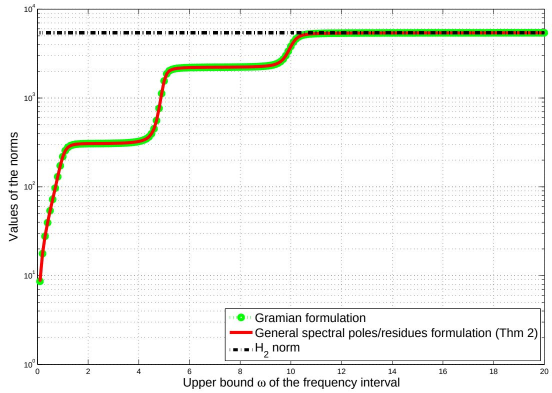

Fig. 3: Frequency-limited $\mathcal{H}_2$-norm of a system with a non-diagonalizable matrix A for $\Omega = [0, \omega]$ with ω going from 0 rad/sec to 8 rad/sec.

be closed under conjugation so that the resulting state-space representation remains real valued.

The frequency-limited $\mathcal{H}_2$-norm of this system is computed for $\Omega = [0, \omega]$ with ω going from 0 to 20 rad/s with both the standard gramian formulation (see Section I) and the poles/residues formulation of Theorem 2. The values of the norms are plotted in Figure 3 with respect to ω together with the $\mathcal{H}_2$-norm of the system.

Both formulations of the $\mathcal{H}_{2,\Omega}$-norm leads to very close results as illustrated in Figure 3. The maximum mismatch error between the two results is in general very small. Large mismatch error might appear as some eigenvalues get closer to the imaginary axis since both formulation of the norm become more ill-conditionned.

IV. Conclusion

In this paper, some results concerning the frequency-limited $\mathcal{H}_2$-norm have been recalled, in particular the poles/residues expression of the $\mathcal{H}_{2,\Omega}$-norm for systems with a diagonalizable matrix A (see Theorem 1). The latter is based on the partial fraction decomposition of the system's transfer function, which is simple in this case. This paper extends this result to the general case of systems with Jordan block of size superior to one and alleviates the assumptions required for the poles/residues formulation of the norm. In the general case, the partial fraction expansion of the transfer function implies more terms thus leading to a more complex, yet more complete, formulation for the $\mathcal{H}_{2,\Omega}$-nom (see Theorem 2). Since the Jordan form of a matrix is very complex to obtain in term of computation, this formulation may be considered mainly as a theoretical tool which offer an alternative expression for the $\mathcal{H}_{2,\Omega}$-norm of system with non-diagonalizable matrix A. In particular, this formulation may be useful in frequency-limited model approximation [19] to understand how the problem is modified when Jordan blocks appear.

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