

Introduction to MORE: a MOdel REduction Toolbox

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Abstract—In many highly technological engineering fields, the use of dedicated computer-based dynamical system modeling software often leads to large dimensional Linear Time Invariant (LTI) models. These kind of models, composed of a large amount of variables might render drastically inefficient many analysis, control design and optimization techniques. As a matter of fact, considerable attention has been devoted to the development of model reduction - or approximation - techniques to eliminate irrelevant state variables. This paper presents a new freely-available MATLAB[®]-based toolbox for approximation of medium and large-scale LTI dynamical models, called MORE (MOdel REduction), which implements a collection of very recent advanced algorithms for LTI dynamical model reduction purpose.

Index Terms—Matlab toolbox, model reduction, large-scale linear dynamical systems.

I. INTRODUCTION

A. Motivation

Both within the industrial and academical areas, dedicated computer-based dynamical system modeling software are ever-increasingly used by engineers and researchers to accurately capture a large variety of complex dynamical phenomena. This is especially true within many highly challenging technological fields, such as industrial flight dynamics, aerospace, automotive, civilian engineering, or for the comprehension of very complex processes such as biological mechanisms, weather forecasting... The counterpart of this faithful modeling is that it often leads to models with a cumbersome number of variables and resources to manage, resulting in a prohibitively expensive numerical cost. Additionally, from the control engineer viewpoint, modern *analysis* and *synthesis* tools (e.g. μ -analysis, LQ, $\mathcal{H}_{\infty,2}$ control...) become drastically inefficient for such high dimensional dynamical systems.

B. Linear projection-based model approximation

Grounded on these observations, a very challenging task consists in developing numerically stable and robust techniques allowing approximating any large-scale dynamical models with lower order ones. This problem has a long lasting history and has been addressed by both the numerical and control communities leading to a wide variety of approaches (see e.g. [1], [2], [3], [4], and references therein). Without loss of generality, the most efficient

medium and large-scale¹ dynamical model approximation techniques, called *projection-based approaches*, are based on the Petrov-Galerkin projection problem. Given a nonlinear dynamical system $\dot{x}(t) = f(x(t), u(t))$ ($x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^{n_u}$), this problem consists in finding projectors (i) $V \in \mathbb{R}^{n \times r}$ ($r \ll n$) forming a basis of subspace \mathcal{V} , and (ii) $W = (\overline{W}^T V)^{-1} \overline{W}$ (where $\overline{W} \in \mathbb{R}^{n \times r}$ is a basis of subspace \mathcal{W}), such that $\overline{W}^T V$ is invertible and

$$\hat{x}(t) \in \mathcal{V} \quad \text{and} \quad \overline{W}^T (V \dot{\hat{x}}(t) - f(V \hat{x}(t), u(t))) = 0 \quad (1)$$

Here $x(t) \approx V \hat{x}(t)$ and $\hat{x}(t) \approx W^T x(t)$ are the full original and reduced order states, respectively. More specifically, within the linear field, the Petrov-Galerkin problem is recast by noting $f(x(t), u(t)) = Ax(t) + Bu(t)$, in (1).

This paper is attached to the *linear problem only*. In this specific case, research can be divided in two mainstreams: (i) the theoretical one, where attention is devoted to the development of new methods guaranteeing some approximation properties (such as stability / passivity, \mathcal{H}_2 optimality...), and, (ii) the numerical one, which focusses on the development of numerically robust and fast procedures (tools) implementing of the aforementioned theoretical methods.

With reference to the first mainstream (i.e. theoretical), a very comprehensive and complete overview of the existing methods can be found in [4], [5]. Within this scope, the following four main families can be highlighted:

- the Lyapunov/Sylvester equations and SVD decomposition methods [2], [6], [7], [8], [9];
- the modal approaches [10], [11];
- the Krylov/Tangential subspace methods [12], [13], [14], [15], [16];
- and the mixed approaches [17], [18].

Similarly, with reference to the second mainstream (i.e. numerical), the following tools can be emphasized:

- the MATLAB[®] Robust Control Toolbox [19], [6], which implements the Gramian and SVD based methods. Since it provides a substantial set of methods, this software will be considered as the reference one in our study;
- the SLICOT [20], which is a MATLAB[®] toolbox implementing numerically reliable and efficient techniques, including Balanced and Truncate singular perturbation approximation, balanced stochastic truncation,

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¹In this paper medium and large-scale models will denote models of order $n < 1000$ and $n < 5000$, respectively.

frequency-weighting balancing... It provides very powerful functions but is, to the authors knowledge, neither longer maintained nor available;

- Another family of tools, based on the modal approaches, is available on J. Rommes personal web page.
- Finally, different specific tools such as MOREMBS (University of Stuttgart), MORPack (TU Dresden), PABTEC, MESS - a Matrix Equation Sparse Solver - successor of Lapack, are also available.

C. Paper contribution, structure & Notations

The contribution of this paper is to introduce MORE (for MModel REDuction), a new simple and user-friendly but very effective MATLAB[®]-based toolbox for Multiple Input Multiple Output (MIMO) Linear Time Invariant (LTI) medium (large)-scale model approximation (see Figure 1). This tool implements very recent developments within the Tangential (and Krylov) family, allowing stability preservation and hopefully reaching the so-called first-order \mathcal{H}_2 optimality conditions. In the authors' view, this kind of tool can be very useful both for industrial or academic researchers and practitioners to approximate some medium and high dimensional problems or to benchmark their reduction techniques with respect to the one presented in this paper.

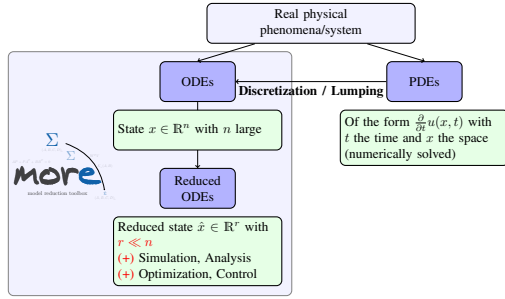


Fig. 1. Reduction chart synopsis and positioning of the MORE toolbox.

The rest of the paper is organized as follows: the projection-based MIMO LTI approximation problem, the so-called \mathcal{H}_2 optimality conditions and a few very recent technical results, mainly based on the tangential approaches, are reminded in Section II. The toolbox and its main routine are presented in Section III. Illustrative examples of the performance and comparison with classical techniques using usual model reduction benchmarks [21] and an industrial flexible aircraft model are presented in Section IV. Conclusion and perspective are gathered in Section V.

Mathematical notations are standard: the original system state is denoted $x \in \mathbb{R}^n$, the reduced-order system one is referred as $\hat{x} \in \mathbb{R}^r$. Full order MIMO LTI state-space (resp. transfer) form is denoted $\Sigma := (A, B, C)$ (resp. $H(s) := C(sI_n - A)^{-1}B$), and the reduced one $\hat{\Sigma} := (\hat{A}, \hat{B}, \hat{C})$ (resp. $\hat{H}(s) := \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$). W and V denote the left and right projectors, respectively. $\mathcal{V} = \text{span}(v)$ means that the space \mathcal{V} is spanned by the column vectors of v . Then, $\lambda(\cdot)$ holds for the eigenvalue operator, $\nabla(\cdot)$ stands for the Gradient operator. The residual of the complex valued function $f(s)$

at λ is denoted $\text{res}(f(s), \lambda) = \lim_{s \rightarrow \lambda} (s - \lambda)f(s)$, where $s = j\omega$ (with $j = \sqrt{-1}$) defines the Laplace variable. The conjugate of v is denoted v^* .

II. PROBLEM STATEMENT, \mathcal{H}_2 OPTIMALITY AND PRELIMINARY RESULTS

A. Problem formulation

The problem, as it is treated in the MORE toolbox, is given as follows.

Problem 1 (MORE toolbox): Let consider a stable and strictly proper n_u inputs n_y outputs MIMO LTI dynamical model $\Sigma := (A, B, C)$ of order n , defined as

$$\Sigma : \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \quad (2)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$ and $C \in \mathbb{R}^{n_y \times n}$. The projection-based approximation problem consists of finding a projector $\Pi_{W,V} = WV^T \in \mathbb{R}^{n \times n}$ (with $V, W \in \mathbb{R}^{n \times r}$, $W^T V = I_r$), such that $\hat{\Sigma} := (\hat{A}, \hat{B}, \hat{C})$, a reduced model of order $r \ll n$, defined as

$$\hat{\Sigma} : \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \quad \hat{y}(t) = \hat{C}\hat{x}(t) \quad (3)$$

where $\hat{A} = W^T A V$, $\hat{B} = W^T B$ and $\hat{C} = C V$, well approximates Σ in the sense of a given measure.

Since the aim of model approximation is to capture the main system dynamics of interest and input/output behaviour, while guaranteeing stability and achieving minimal model mismatch, the \mathcal{H}_2 optimal approximation problem is often addressed. It consists of seeking an approximation $\hat{H}(s)$ of $H(s)$, such that,

$$\min_{\hat{H}(s) \text{ (stable, order } r)} \|H(s) - \hat{H}(s)\|_{\mathcal{H}_2} \quad (4)$$

Reader may note that attention to the \mathcal{H}_∞ error should also be paid (since it provides the worst case error). In a similar way, the \mathcal{H}_∞ approximation problem consists of seeking $\hat{H}(s)$ such that,

$$\min_{\hat{H}(s) \text{ (stable, order } r)} \|H(s) - \hat{H}(s)\|_{\mathcal{H}_\infty} \quad (5)$$

The latter problem (5) is practically very complex to achieve for large (medium)-scale models due to the iterative nature of the \mathcal{H}_∞ norm computation, while, even if it is non-convex, *first-order optimality conditions* of the former one (4) can practically be characterized and satisfied (see [7], [4], [16], [22], [8]).

Remark 1: Since the \mathcal{H}_2 and \mathcal{H}_∞ norms are defined only for stable systems, when one aim at approximating an unstable system, the stable part only is reduced while the unstable one is kept as it is. An other approach is to define a shifted \mathcal{H}_2 and \mathcal{H}_∞ norms.

B. \mathcal{H}_2 first order optimality results

The \mathcal{H}_2 norm of a n -th order strictly proper stable system $H(s)$ with simple poles at $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ is expressed as

$$\begin{aligned} \|H(s)\|_{\mathcal{H}_2}^2 &= \text{trace}(B^T Q B) = \text{trace}(C P C^T) \\ &= \sum_{i=1}^n \text{res}(H^T(s), \lambda_i) H(-\lambda_i) \end{aligned} \quad (6)$$

where Q and P are the observability and controllability Gramian. Based on the first line of (6), the following theorem/corollary hold (essentially due to [1]).

Theorem 1 (Wilson \mathcal{H}_2 first-order optimality conditions): Given $H(s) := C(sI_n - A)^{-1}B$, $\hat{H}(s) := \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$, the controllability and observability Gramian $\mathcal{P}_e = \begin{bmatrix} P & X \\ X^T & \hat{P} \end{bmatrix}$ and $\mathcal{Q}_e = \begin{bmatrix} Q & Y \\ Y^T & \hat{Q} \end{bmatrix}$, of the error system $H(s) - \hat{H}(s)$. If, $\hat{Q}\hat{P} + Y^T X = 0$, $\hat{Q}\hat{B} + Y^T B = 0$ and $\hat{C}\hat{P} - CX = 0$, then (with $\mathcal{J} = \|H(s) - \hat{H}(s)\|_{\mathcal{H}_2}$)

$$\nabla_{\hat{A}}\mathcal{J} = 0 \quad , \quad \nabla_{\hat{B}}\mathcal{J} = 0 \quad \text{and} \quad \nabla_{\hat{C}}\mathcal{J} = 0 \quad , \quad (7)$$

The reduced model is then obtained by linking the projectors and the stationary conditions as follows.

Corollary 1 (Wilson \mathcal{H}_2 optimality conditions): At every stationary point of functional \mathcal{J} (*i.e.* $\nabla\mathcal{J} = 0$) where \hat{P} and \hat{Q} are invertible, we have the following identities: $\hat{A} = W^T A V$, $\hat{B} = W^T B$ and $\hat{C} = C V$ with $W^T V = I_r$, $W = -Y\hat{Q}^{-1}$, $V = X\hat{P}^{-1}$ and where X , Y , \hat{P} and \hat{Q} satisfies the error system $H(s) - \hat{H}(s)$ Sylvester equations,

$$\begin{aligned} AP + PA^T + BB^T &= 0 & QA + A^T Q + C^T C &= 0 \\ \hat{A}X^T + X^T A^T + \hat{B}B^T &= 0 & A^T Y + Y \hat{A} - C^T \hat{C} &= 0 \\ \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T &= 0 & \hat{Q}\hat{A} + \hat{A}^T \hat{Q} + \hat{C}^T \hat{C} &= 0 \end{aligned} \quad (8)$$

Based on the second line of (6), similar results can be derived (see contributive works of [14], [16]). More specifically, the following theorem/corollary hold.

Theorem 2 (Tangential \mathcal{H}_2 optimality conditions): If $\nabla_{\hat{A}}\mathcal{J} = 0$, $\nabla_{\hat{B}}\mathcal{J} = 0$ and $\nabla_{\hat{C}}\mathcal{J} = 0$, then the tangential interpolation conditions are satisfied for all $\hat{\lambda}_i$, $i = 1, \dots, r$

$$\begin{aligned} [H(-\hat{\lambda}_i) - \hat{H}(-\hat{\lambda}_i)]\hat{b}_i &= 0 \quad , \quad \hat{c}_i^*[H(-\hat{\lambda}_i) - \hat{H}(-\hat{\lambda}_i)] = 0 \\ \hat{c}_i^* \frac{d}{ds} [H(s) - \hat{H}(s)]|_{s=-\hat{\lambda}_i} \hat{b}_i &= 0 \end{aligned} \quad (9)$$

where $\{\hat{b}_1, \dots, \hat{b}_r\} = \hat{B}^T R$ and $\{\hat{c}_1, \dots, \hat{c}_r\} = \hat{C} L$ (L and R are the left and right eigenvectors associated to $\hat{\lambda}$, the eigenvalues of \hat{A}).

Corollary 2 shows how to construct the projectors V and W to fulfil any tangential interpolations [15], [16].

Corollary 2 (Tangential \mathcal{H}_2 optimality conditions): Let $V \in \mathbb{C}^{n \times r}$ and $W \in \mathbb{C}^{n \times r}$ be matrices of full rank r such that $W^T V = I_r$. Let $\sigma_i \in \mathbb{C}^r$, $\hat{b}_i \in \mathbb{C}^{n_u}$ and $\hat{c}_i \in \mathbb{C}^{n_y}$ (for $i = 1, \dots, r$) be given sets of interpolation points and left and right tangential directions, respectively. Assume that points σ_i are selected such that $\sigma_i I_n - A$ are invertible. If, for $i = 1, \dots, r$, $(\sigma_i I_n - A)^{-1} B \hat{b}_i \in \text{span}(V)$ and $(\sigma_i I_n - A^T)^{-1} C^T \hat{c}_i \in \text{span}(W)$, then, the reduced order system \hat{H} , satisfies the tangential interpolation conditions,

$$\begin{aligned} H(\sigma_i)\hat{b}_i &= \hat{H}(\sigma_i)\hat{b}_i \quad , \quad \hat{c}_i^* H(\sigma_i) = \hat{c}_i^* \hat{H}(\sigma_i) \\ \hat{c}_i^* \frac{d}{ds} H(\sigma_i)\hat{b}_i &= \hat{c}_i^* \frac{d}{ds} \hat{H}(\sigma_i)\hat{b}_i \end{aligned} \quad (10)$$

Remark 2: In two very recent works, [8] and [22], the Wilson conditions of Theorem 1 and the tangential interpolation based optimality conditions of Theorem 2 have been shown to be equivalent.

Since these theorems only provide a characterisation and an equivalence of the first-order optimality conditions from either a Sylvester or a Tangential point of view, the challenge consists in creating procedures achieving these conditions. More specifically, based on Corollary 2, the \mathcal{H}_2 optimal problem consists of finding the triplet $\{\sigma_i, \hat{b}_i, \hat{c}_i\}$ to satisfy Theorem 2. With reference to Corollary 1, similar investigations focus on the Gramian computations, *i.e.*, finding the quadruplet $\{\hat{P}, \hat{Q}, X, Y\}$.

C. Model approximation results

To reach the first-order optimality conditions presented above, considerable attention has been addressed to three families of methods, (*i*) the Krylov/Tangential, (*ii*) the Sylvester/Lyapunov and (*iii*) the mixed ones. Considering the former one:

- The Iterative Rational Krylov Algorithm (IRKA), initially set for SISO systems [15], allows satisfying the \mathcal{H}_2 first-order optimality conditions but did not a priori preserves stability. Later, in [23], authors extended it to MIMO systems, with a complex Trust Region algorithm, guaranteeing \mathcal{H}_2 mismatch error monotonic descent and stability preservation.
- Almost at the same period, the Iterative Tangential Interpolation Algorithm (ITIA), has been suggested in the works of [16], [22]. It provides substantial results in the MIMO case. As the previous one, this procedure shows to be effective in many classical benchmark [21] but does not preserve stability, a priori. The ITIA (or equivalently MIMO IRKA) is recalled in Algorithm 1. This algorithm is implemented in the MORE toolbox (`moreLTI(sys, r, 'ITIA')`), see Section III).

On the other hand, Sylvester based approaches have been developed to approximate MIMO LTI systems *e.g.*:

- The Balanced Truncation (BT), which is often considered as the "gold standard" since it preserves stability, provides a (even pessimistic) bound on the \mathcal{H}_∞ error and a nearly optimal \mathcal{H}_2 mismatch error [19], [4]. One of the main drawback of this kind of approach is that it may practically fail when $n > 1000$.
- The Two-Sided Iterative Algorithm (TSIA) [8], which iteratively solves two Sylvester equations, is shown to be equivalent to the tangential interpolation. This procedure guarantees stability but suffers of two main drawbacks: first, it requires a good projector initialisation (*i.e.* V and W) to converge, and secondly, no stopping criterion is given. Recently, authors of [9] explore three different stopping criteria that can be used for this algorithm. This algorithm will be available future versions of the MORE toolbox.

As grounded on the two previous families, due to [17], a new family, called mixed approaches, where both Gramian and Tangential approaches, have merged. Within this field, the following methods can be highlighted:

- In [17], author present an Iterative SVD Rational Krylov Algorithm (ISRKA), allowing one sided moments matching and stability preservation by mean of

Algorithm 1 Iterative Tangential Interpolation Algorithm (ITIA) - `moreLTI(sys, r, 'ITIA')`

Require: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$, $\{\sigma_1^{(0)}, \dots, \sigma_r^{(0)}\} \in \mathbb{C}^{n \times r}$, $\{\hat{b}_1, \dots, \hat{b}_r\} \in \mathbb{C}^{n \times r}$, $\{\hat{c}_1, \dots, \hat{c}_r\} \in \mathbb{C}^{n \times r}$, $\varepsilon > 0$

1: Construct,

$$\begin{aligned} \text{span}(V) &= [(\sigma_1^{(0)}I - A)^{-1}B\hat{b}_1, \dots, (\sigma_r^{(0)}I - A)^{-1}B\hat{b}_r] \\ \text{span}(W) &= [(\sigma_1^{(0)}I - A)^{-T}C^T\hat{c}_1, \dots, (\sigma_r^{(0)}I - A)^{-T}C^T\hat{c}_r] \end{aligned}$$

2: Compute $W \leftarrow W(V^T W)^{-1}$

3: **while** $|\sigma^{(i)} - \sigma^{(i-1)}| > \varepsilon$ **do**

4: $i \leftarrow i + 1$, $\hat{A} = W^T A V$, $\hat{B} = W^T B$ and $\hat{C} = C V$

5: Compute $\hat{A}X = \text{diag}(\lambda(\hat{A}))X$

6: Compute $\{\hat{b}_1, \dots, \hat{b}_r\} = X^{-1}\hat{B}$, $\{\hat{c}_1, \dots, \hat{c}_r\} = \hat{C}X$

7: Set $\sigma^{(i)} = -\lambda(\hat{A})$

8: Construct,

$$\begin{aligned} \text{span}(V) &= [(\sigma_1^{(i)}I - A)^{-1}B\hat{b}_1, \dots, (\sigma_r^{(i)}I - A)^{-1}B\hat{b}_r] \\ \text{span}(W) &= [(\sigma_1^{(i)}I - A)^{-T}C^T\hat{c}_1, \dots, (\sigma_r^{(i)}I - A)^{-T}C^T\hat{c}_r] \end{aligned}$$

9: Compute $W \leftarrow W(V^T W)^{-1}$

10: **end while**

Ensure: $W^T V = I_r$

the computation of one single Gramian. The approach handles SISO, SIMO and MISO models.

- In [18], the previous algorithm has been extended with tangential interpolation allowing to tackle the MIMO models. The procedure, called Iterative SVD-Tangential Krylov Algorithm (ISTIA), is recalled in Algorithm 2 and is implemented in the MORE toolbox (`moreLTI(sys, r, 'ISTIA')`, see Section III).

III. DESCRIPTION OF THE TOOLBOX

A. Brief overview, features and first use

The actual freely available MORE toolbox version (working with MATLAB[®] 2010a and higher) is accessible at <http://www.onera.fr/staff-en/charles-poussot-vassall/>. This version allows to approximate SISO and MIMO LTI models with up to $n = 50$ states. In its full version, accessible upon request, the toolbox can handle at least $n = 3000$ states. The toolbox is composed of two directories: (i) the 'examples' one, which gathers some demo files and (ii) the 'routines' one, which contains the main routines and the sub-routines of the toolbox. Additionally the root folder contains the standard content file and the MORE logo.

Remark 3: In order to be used for benchmarking and academical research purpose, some models contained in COMPl_eib benchmark [21] - of higher order - can also be approximated (the COMPl_eib toolbox must be installed).

B. The `moreLTI` routine

As illustrated in the next sections, the toolbox implements one main user-friendly interface, called `moreLTI`. As described in Table I, this function implements two

Algorithm 2 Iterative SVD-Tangential Interpolation Algorithm (ISTIA) - `moreLTI(sys, r, 'ISTIA')`

Require: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$, $\{\sigma_1^{(0)}, \dots, \sigma_r^{(0)}\} \in \mathbb{C}^{n \times r}$, $\{\hat{b}_1, \dots, \hat{b}_r\} \in \mathbb{C}^{n \times r}$, $\varepsilon > 0$

1: Construct,

$$\text{span}(V) = [(\sigma_1^{(0)}I - A)^{-1}B\hat{b}_1, \dots, (\sigma_r^{(0)}I - A)^{-1}B\hat{b}_r]$$

2: Solve $QA + A^T Q + C^T C = 0$ in Q

3: Compute $W = QV(V^T QV)^{-1}$

4: **while** $|\sigma^{(i)} - \sigma^{(i-1)}| > \varepsilon$ **do**

5: $i \leftarrow i + 1$, $\hat{A} = W^T A V$, $\hat{B} = W^T B$

6: Compute $\hat{A}X = \text{diag}(\lambda(\hat{A}))X$

7: Compute $\{\hat{b}_1, \dots, \hat{b}_r\} = X^{-1}\hat{B}$

8: Set $\sigma^{(i)} = -\lambda(\hat{A})$

9: Construct,

$$\text{span}(V) = [(\sigma_1^{(i)}I - A)^{-1}B\hat{b}_1, \dots, (\sigma_r^{(i)}I - A)^{-1}B\hat{b}_r]$$

10: Compute $W = QV(V^T QV)^{-1}$

11: **end while**

Ensure: $W^T V = I_r$, $\text{Re}(\sigma^{(i)}) < 0$

main algorithms: (i) Algorithm 1, which might be called as `moreLTI(sys, r, 'ITIA')` and (ii) Algorithm 2, which might be called as `moreLTI(sys, r, 'ISTIA')`.

Remark 4: With reference to Algorithm 1 and 2, initial shift selection $\{\sigma_1^{(0)}, \dots, \sigma_r^{(0)}\}$ have to be done by the user. In the toolbox, this tricky choice can be alleviated by mean of an automatic choice using the procedure proposed in [24].

Remark 5: Note that the `moreLTI` function will be in constant evolution, *i.e.* will implement many other reduction techniques.

IV. EXAMPLE AND ILLUSTRATION ON USE CASES

To benchmark the MORE toolbox with respect to other well established methods (*e.g.* the MATLAB[®] `reduce` routine), the `moreLTI` routine with different options are applied on classical benchmarks extracted from [21], namely, the SISO Los-Angeles Hospital ('LAH') and the SISO beam models ('CBM'). Additionally, the proposed benchmark also includes a MIMO industrial aircraft model, in order to assess the toolbox on very realistic badly damped and ill conditioned industrial problem. To evaluate the performance, the following relative \mathcal{H}_2 error metrics will be considered:

$$\varepsilon_{\mathcal{H}_2} = \frac{\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_2}}{\|\Sigma\|_{\mathcal{H}_2}} \quad (11)$$

Remark 6: All results have been achieved on a standard computer, *i.e.* Intel(R) Core(TM)2 Duo CPU, 3GHz.

A. Los-Angeles Hospital ('LAH', $n = 48$, $n_u = 1$, $n_y = 1$)

A first evaluation of the MORE toolbox can be achieved by evaluating the following sequence:

```
>> [A,~,B,~,C] = COMPl_eib('LAH');
>> sys = ss(A,B,C,0); r = 14;
>> sysBT = reduce(sys,r);
```

Input			
sys	Original LTI system	ss, string (COMPl _e ib) or matrix list $\{A, B, C\}$	'LAH', 'CDP', 'CBM', 'ISS1', 'CM4', 'CM5', 'CM6',
r	Reduction order	$\in \mathbb{N}^*$	
method	Method selection	'ITIA' or 'ISTIA' (default 'ISTIA')	
opt	Optional arguments	struct	
↪ shiftPoint $\sigma^{(0)}$	Initial interpolation points	$\in \mathbb{C}^{r/2}$ (default, [24] selection)	
↪ tol ϵ	Tolerance error	$\in \mathbb{R}^{+*}$ (default, 0.1)	
↪ maxIter	Maximum number of iterations	$\in \mathbb{N}^*$ (default, 30)	
↪ restart	Number of restarts	$\in \mathbb{R}^{+*}$ (default, 2)	
↪ display	Iteration information	1 (display on), 0 (default, display off)	
↪ gram	Computed Gramian	'o' or 'c' (default, 'o')	only with 'ISTIA'
Output			
sysRed	Reduced order model	ss, order r	
out	Output parameters	struct	
↪ V	Right projector	$\in \mathbb{R}^{n \times r}$	
↪ W	Left projector	$\in \mathbb{R}^{n \times r}$	
↪ shiftPoint $\sigma^{(N)}$	Final interpolation points	$\in \mathbb{C}^r$	
↪ iterOffset N	Number of iterations	$\in \mathbb{N}^{+*}$	
↪ absH2err	Absolute \mathcal{H}_2 error	$\in \mathbb{R}^{+*}$	if opt.restart > 0

TABLE I

INPUT / OUTPUT PARAMETERS OF THE ITIA AND ISTIA: $[\text{sysRed}, \text{out}] = \text{moreLTI}(\text{sys}, r, \text{method}, \text{opt})$;

```
>> opt.restart = 0;
>> [sysITIA, out] = moreLTI(sys, r, 'ITIA', opt);
>> [sysISTIA, out] = moreLTI(sys, r, 'ISTIA', opt);
```

Within few seconds, the following relative \mathcal{H}_2 norm error (11) are obtained: (i) 1.456456×10^{-1} with the MATLAB[®] reduce, (ii) 1.135438×10^{-1} with 'ITIA' method, and (iii) 1.15588×10^{-1} with the 'ISTIA' one. At this step, interested user may also notice that $W^T V \approx I_r$ and that the structure out contains the final shifts which correspond the optimal selection (out.shiftPoint) and the number of performed iterations (out.iterOffset).

Remark 7: These results can be obtained running demoMORE0. A similar example, with a larger MIMO CD player model is available using demoMORE1.

B. Clamped beam ('CBM', $n = 348$, $n_u = 1$, $n_y = 1$)

Let consider the clamped beam model using default values and no restart (i.e. opt.restart=0). The approximation is performed for varying reduction order r . Top frame of Figure 2 illustrates the mismatch error, showing the very good results of the MORE toolbox with respect to the standard reduction MATLAB[®] function (i.e. reduce). The bottom frame of Figure 2 shows the computational time of all methods. It emphasizes the fact that the ITIA outperforms both methods on the speed factor while the ISTIA is much lower, due to the Gramian computation. Anyhow, both MORE methods provides better results than the standard reduce function in very few seconds (run demoMORE2).

C. Industrial flexible aircraft ($n = 540$, $n_u = 2$, $n_y = 4$)

Here, a real life MIMO industrial longitudinal aeroelastic model, used by engineers both for control synthesis and performance analysis, is considered. Approximating such system is a very challenging and time consuming task since the conditioning number is very high and numerous badly damped mode are present (see e.g. [18], [25]). As

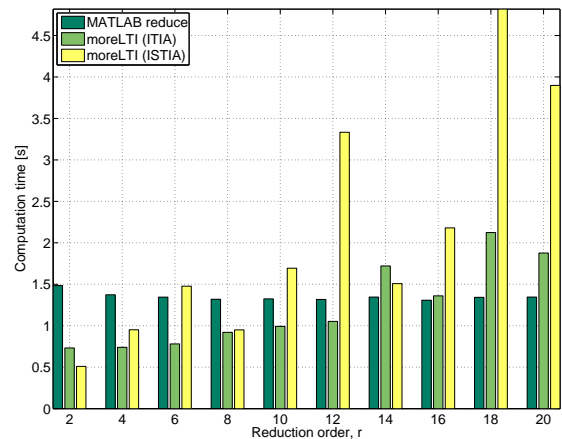
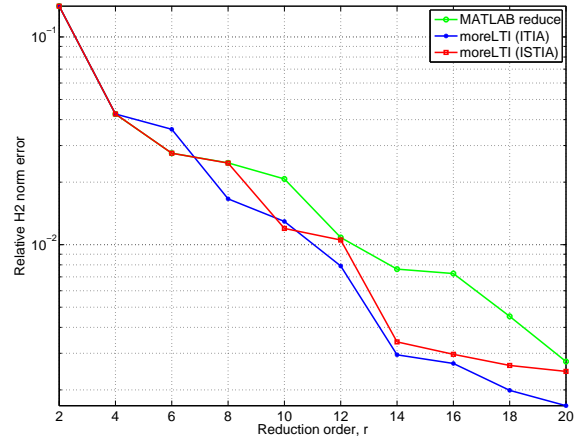


Fig. 2. Model relative \mathcal{H}_2 mismatch error (top) and computational time (bottom), applied to the beam model.

in the previous subsection, the two approximation routines proposed in the MORE toolbox are compared to the best approximation result provided by the MATLAB[®] reduction routine. Figure 3 illustrates the approximation error results as a function of the reduction order r , emphasizing the efficiency of the proposed toolbox.

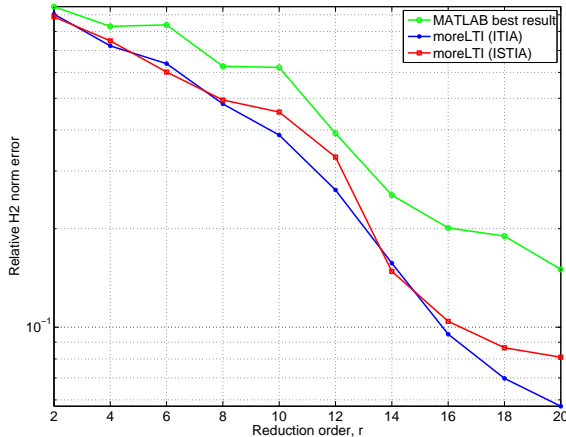


Fig. 3. Model mismatch \mathcal{H}_2 error of the industrial aircraft model.

V. CONCLUSION AND PERSPECTIVES

In this paper, MORE, a new toolbox dedicated to the approximation of MIMO LTI medium (large)-scale dynamical systems, has been presented. To this purpose, the toolbox implements very recent theoretical results by mean of two user-friendly interfaces. Comparisons with classical approximation techniques have shown very positive results both on classical reduction benchmark models and on a complex industrial one, making this new tool very appealing for practitioners and researchers. The toolbox aims to be maintained and enhanced. Forthcoming improvements will include (i) velocity enhancement, (ii) initial shift selection improvement, (iii) implementation of Sylvester like approaches [8], (iv) multi-model reduction [25], (v) implementation of different stopping criteria [9] and (vi) frequency limited reduction procedures. The tool is available on the authors institutional page (in limited use), or upon request (full version).

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