Approximation of stability regions for large-scale time-delay systems using model reduction techniques

I. Pontes Duff, P. Vuillemin, C. Poussot-Vassal, C. Seren and C. Briat

***Abstract*— In this paper, the problem of determining the approximate stability regions of large-scale time-delay systems (LS TDS) is solved using model approximation techniques. To achieve this, an $\mathcal{H}_2$-oriented approximation algorithm, referred to as TF-IRKA [1], is considered. This algorithm has been shown to be well suited for the approximation of infinite-dimensional systems into finite-dimensional ones. We show here how model reduction can be used to approximate time-delay systems with multiple delays and estimate their stability regions. Discussions regarding the adaptation of existing algorithms to the considered problem are also provided. Several numerical examples illustrate the efficiency and accuracy of the approach.**

***Index Terms*— Model approximation; large-scale systems; time-delay systems**

I. Introduction

Time-delay systems (TDS) is a broad class of systems that can model a wide range of real world systems and phenomena [2]. Linear time-delay systems with discrete-delays can be expressed as

$$\dot{x}(t) = Ax(t)+\sum_{i=1}^{n_d} A_i x(t-\tau_i)+Bu(t),\ y(t) = Cx(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^{n_u}$ is the input, $y(t) \in \mathbb{R}^{n_y}$ is the output and $\tau_i \in \mathbb{R}_+$, $i = 1,\ldots,n_d$, are the constant delays. It is known that time-delay systems have an infinite number of eigenvalues and that their corresponding transfer functions are irrational. This is the reason why many tailored approaches have been developed to characterize the properties of time-delay systems such as stability; see *e.g.* [3], [4]. However, in many practical situations, the stability conditions cannot be analytically checked since they cannot be solved explicitly. In this respect, alternative techniques relying on numerical calculations, such as optimization-based methods, have to be considered [4]. Unfortunately, when the dimension of the state vector $x(t) \in \mathbb{R}^n$ becomes large (*e.g.* $n \gg 100$ as in [5], Example 4 below), many of the existing numerical methods, such as those based on linear matrix inequalities, become intractable.

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In order to simplify the model, model-reduction techniques, such as those proposed in [6], [7], can be considered. Yet, in these cases, the reduced model is also a time-delay system. The approach considered here aims at determining a finite-dimensional system (*i.e.* without delay) that approximates the original model using optimal $\mathcal{H}_2$-oriented model reduction techniques. Such model reduction methods have been proven to be adapted to the reduction of large-scale systems as a preliminary step before their analysis and control; see [8]–[11]. To this aim, we propose to use optimal $\mathcal{H}_2$-oriented realization-independent model approximation methods [1] for approximating large-scale time-delay systems with multiple delays in view of establishing interesting properties of the original model from the reduced one. The rationale is that the combined computational cost of the reduction and the analysis of the properties of the reduced model is drastically lower than by dealing directly with the original model.

Notations. We denote by $\mathbb{N}^*$ the set of positive integers. We denote by $\mathcal{L}_2(i\mathbb{R})$ the Hilbert space of matrix-valued functions $F : \mathbb{C} \to \mathbb{C}^{n_y\times n_u}$ satisfying $\int_{\mathbb{R}} \mathbf{trace}[\overline{F(i\omega)}F(i\omega)^T]d\omega < \infty$. We say that a linear time-invariant system $\mathbf{H}$ is in $\mathcal{L}_2(i\mathbb{R})$ if its transfer function $H(s)$ has no pole on the imaginary axis. For $\mathbf{H}, \mathbf{G} \in \mathcal{L}_2(i\mathbb{R})$, we define the inner-product

$$\langle \mathbf{H}, \mathbf{G}\rangle_{\mathcal{L}_2} = \int_{-\infty}^{\infty} \mathbf{trace}\Big(\overline{H(i\omega)}G(i\omega)^T\Big)d\omega,$$

with corresponding induced-norm $\|\mathbf{H}\|_{\mathcal{L}_2} = \langle \mathbf{H}, \mathbf{H}\rangle_{\mathcal{L}_2}^{\frac{1}{2}}$. Let $\mathcal{H}_2(\mathbb{C}^+)$ be the closed subspace of $\mathcal{L}_2(i\mathbb{R})$ which contains the matrix functions $F(s)$ analytic in the open right-half plane and $\mathcal{H}_2(\mathbb{C}^-)$, the closed subspace of $\mathcal{L}_2(i\mathbb{R})$ which contains the matrix functions $F(s)$ analytic in the open left-half plane. Note that $\mathcal{L}_2(i\mathbb{R}) = \mathcal{H}_2(\mathbb{C}^-)\bigoplus\mathcal{H}_2(\mathbb{C}^+)$. According to these definitions, the set of stable LTI systems is $\mathcal{H}_2(\mathbb{C}^+)$ while the set of unstable LTI systems is characterized by $\mathcal{L}_2(i\mathbb{R})\backslash\mathcal{H}_2(\mathbb{C}^+)$, where $\backslash$ denotes the set-difference. Finally, we denote by (E, A, B, C) the finite-dimensional linear time-invariant system given by

$$E\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t). \quad (2)$$

Outline. The paper is organized as follows. Section II recalls preliminary results on realization-independent model approximation as well as some rational interpolation results from the literature. Section III presents the main results of this paper, namely arguments for stability approximation and an adaption of **TF-IRKA** to time-delay systems. Finally, Section IV illustrates the method through several examples.

II. Preliminary results in model approximation

A. Optimal $\mathcal{H}_2$ and $\mathcal{L}_2$ approximation problems

The optimal $\mathcal{H}_2$ model approximation problem is stated as follows [12], [13].

Problem 1 ($\mathcal{H}_2$ model approximation): Given an LTI system $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$ and an $r \in \mathbb{N}^*$, find a finite-dimensional stable LTI system $\hat{\mathbf{H}}$ such that

$$\hat{\mathbf{H}} := \underset{\mathbf{G}\in\mathcal{H}_2(\mathbb{C}^+),\dim(\mathbf{G})\leq r}{\operatorname{argmin}} \|\mathbf{H}-\mathbf{G}\|_{\mathcal{H}_2}. \tag{3}$$

This problem is nonconvex and may be difficult to solve for high dimensional systems. However, it as been pointed out over the last decade that when a realization $\mathbf{H} = (E, A, B, C)$ exists, many numerically efficient procedures can be used, such as **IRKA** [14], [15], **DARPO** [16] and **ISTIA** [9], [17].

The above problem can be generalized to unstable systems as follows [18].

Problem 2 ($\mathcal{L}_2$ model approximation): Given an LTI system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$ and an $r \in \mathbb{N}^*$, find a finite dimensional LTI system $\hat{\mathbf{H}}$ such that

$$\hat{\mathbf{H}} := \underset{\mathbf{G}\in\mathcal{L}_2(i\mathbb{R}),\dim(\mathbf{G})\leq r}{\operatorname{argmin}} \|\mathbf{H}-\mathbf{G}\|_{\mathcal{L}_2}. \tag{4}$$

B. First-order $\mathcal{H}_2$ and $\mathcal{L}_2$ optimality conditions

Let recall here the first-order optimality conditions for Problem 1 and Problem 2.

Proposition 1 ($\mathcal{H}_2$-optimal model approximation [14]): Assume that $\mathbf{H}$ and $\hat{\mathbf{H}}$ have semi-simple poles and suppose that $\hat{\mathbf{H}}$ is a r^{th}-order finite-dimensional model with transfer function

$$\hat{H}(s) = \sum_{k=1}^{r} \frac{\hat{c}_k \hat{b}_k^T}{s-\hat{\lambda}_k}. \tag{5}$$

If $\mathbf{H}, \hat{\mathbf{H}} \in \mathcal{H}_2$ and $\hat{\mathbf{H}}$ is a local minimum of the $\mathcal{H}_2$ approximation problem (3), then the following interpolations equations hold

$$H(-\hat{\lambda}_k)\hat{b}_k = \hat{H}(-\hat{\lambda}_k)\hat{b}_k,\ \hat{c}_k^T H(-\hat{\lambda}_k) = \hat{c}_k^T \hat{H}(-\hat{\lambda}_k) \tag{6}$$

$$\hat{c}_k^T \left.\frac{dH}{ds}\right|_{s=-\hat{\lambda}_k} \hat{b}_k = \hat{c}_k^T \left.\frac{d\hat{H}}{ds}\right|_{s=-\hat{\lambda}_k} \hat{b}_k, \tag{7}$$

for all $k = 1, \ldots, r$ where $\hat{\lambda}_k$ are the poles of $\hat{\mathbf{H}}$ and $\hat{b}_k$ and $\hat{c}_k$ are its tangential directions, respectively.

Relations (6) and (7) are celebrated as the first order $\mathcal{H}_2$ optimality conditions and state that a local minimum is a bitangential Hermite interpolant of the original model evaluated at the mirror images of the low-order model poles with respect to its tangential directions. In the case where $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$ is a SISO LTI system and $\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^-$ where $\mathbf{H}^+ \in \mathcal{H}(\mathbb{C}^+)$ and $\mathbf{H}^- \in \mathcal{H}(\mathbb{C}^-)$, it is possible to state the following result:

Proposition 2 ($\mathcal{L}_2$ model approximation [18]): Given $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$ and its decomposition $\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^-$ where $\mathbf{H}^+ \in \mathcal{H}(\mathbb{C}^+)$ and $\mathbf{H}^- \in \mathcal{H}(\mathbb{C}^-)$. Let $\hat{\mathbf{H}}$ be the local minimizer of order r whose poles are all simple $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_k\} \in \mathbb{C}^-$ and $\{\hat{\lambda}_{k+1}, \ldots, \hat{\lambda}_r\} \in \mathbb{C}^+$ $(k < r)$. If $\hat{H}(s)$ is given as (5) and if it is a local minimal of the $\mathcal{L}_2$ approximation problem, the following holds for $i = 1, \ldots, k$

$$H^+(-\hat{\lambda}_i) = \hat{H}^+(-\hat{\lambda}_i),\ \left.\frac{dH^+}{ds}\right|_{s=-\hat{\lambda}_i} = \left.\frac{d\hat{H}^+}{ds}\right|_{s=-\hat{\lambda}_i} \tag{8}$$

and for $i = k+1, \ldots, r$,

$$H^-(-\hat{\lambda}_i) = \hat{H}^-(\hat{\lambda}_i),\ \left.\frac{dH^-}{ds}\right|_{s=-\hat{\lambda}_i} = \left.\frac{d\hat{H}^-}{ds}\right|_{s=-\hat{\lambda}_i}. \tag{9}$$

In both the $\mathcal{H}_2$ and $\mathcal{L}_2$ cases, if a finite realization $\mathbf{H} = (E, A, B, C)$ exists, many projection and interpolatory methods enables the construction of such an $\hat{\mathbf{H}}$ interpolant; see [8], [19], [20] for more details.

C. Realization-independent $\mathcal{H}_2$ approximation

In the infinite-dimensional case, the decomposition of $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$ into $\mathcal{H}_2(\mathbb{C}^-)\bigoplus\mathcal{H}_2(\mathbb{C}^+)$ is a difficult task. In this respect, the $\mathcal{L}_2$ optimality conditions (8) and (9) are delicate to apply. Thus, from now on we will be interested to find finite dimensional systems satisfying the $\mathcal{H}_2$ optimality conditions (6) and (7). Moreover, finding a realization of the system is not always possible. This has motivated the introduction of interpolation methods for realization-less models for systems where only the transfer function $H(s)$ is available [21].

Theorem 1 (Interpolatory Loewner framework [21]): Given a system represented by its transfer function $H(s)$, r shifts points $\{s_1, \ldots, s_r\} \in \mathbb{C}$ and r tangential directions $\{c_1, \ldots, c_r\} \in \mathbb{C}^{n_y\times 1}$, $\{b_1, \ldots, b_r\} \in \mathbb{C}^{n_u\times 1}$, the r-dimensional descriptor model $\hat{\mathbf{H}} = (\hat{E}, \hat{A}, \hat{B}, \hat{C})$, as in (2), interpolates $H(s)$ as in (6)-(7) if for $i, j = 1, \ldots, r$,

$$(\hat{E})_{ij} = \begin{cases} -\dfrac{c_i^T(H(s_i)-H(s_j))b_j}{s_i-s_j} & i \neq j \\ -c_i^T H'(s_i) b_i & i = j \end{cases}$$

$$(\hat{A})_{ij} = \begin{cases} -\dfrac{c_i^T(s_iH(s_i)-s_jH(s_j))b_j}{s_i-s_j} & i \neq j \\ -c_i^T (sH(s))'|_{s=s_i} b_i & i = j \end{cases}$$

$$\begin{aligned} \hat{C} &= [H(s_1)b_1, \ldots, H(s_r)b_r] \text{ and} \\ \hat{B}^T &= [H(s_1)^T c_1, \ldots, H(s_r)^T c_r]. \end{aligned} \tag{10}$$

Consequently, the resulting reduced-order model $\hat{H}(s) = \hat{C}(s\hat{E}-\hat{A})^{-1}\hat{B}$ satisfies, for $k = 1, \ldots r$,

$$\begin{aligned} H(s_k)b_k = \hat{H}(s_k)b_k,\ c_k^T H(s_k) = c_k^T \hat{H}(s_k) \\ c_k^T \left.\frac{dH}{ds}\right|_{s=s_k} b_k = c_k^T \left.\frac{d\hat{H}}{ds}\right|_{s=s_k} b_k \end{aligned} \tag{11}$$

at the given complex interpolation points $\{s_1, \ldots, s_r\}$, and tangential directions $\{c_1, \ldots c_r\}$ and $\{b_1, \ldots, b_r\}$. To obtain the optimal approximation satisfying (6) and (7), one must seek for the triplet $\{\hat{\lambda}_i,\ \hat{b}_i, \hat{c}_i\}$. As they are not a priori known, the authors of [1] propose an iterative algorithm which allows to find suitable triplets and, thus, a finite-dimensional system that satisfies the optimal conditions. This procedure, known as **TF-IRKA**, is recalled in Algorithm 1.

Algorithm 1 TF-IRKA [1]

1: **Initialization:** transfer function $H(s)$, $r \in \mathbb{N}^*$, $\sigma^0 = \{\sigma_1^0, \ldots, \sigma_r^0\} \in \mathbb{C}$ initial interpolation points and tangential directions $\{b_1, \ldots, b_r\} \in \mathbb{C}^{n_u \times 1}$, $\{c_1, \ldots, c_r\} \in \mathbb{C}^{n_y \times 1}$.
2: **while** not convergence **do**
3: **Build** $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$ using Theorem 1.
4: Solve the generalized eigenvalue problem $\hat{A}^{(k)} x_i^{(k)} = \lambda_i^{(k)} \hat{E}^{(k)} x_i^{(k)}$ where $y_i^{(k)*} \hat{E}^{(k)} x_j^{(k)} = \delta_{i,j}$.
5: *Set* $\sigma_i^{(k+1)} \leftarrow -\lambda_i^{(k)}$, $b_i^{(k+1)T} \leftarrow y_i^{(k)} \hat{B}^{(k)}$ and $c_i^{(k+1)} \leftarrow \hat{C}^{(k)} x_i^{(k)}$, for $i = 1, \ldots, r$.
6: **end while**
7: **Ensure** conditions (6) are satisfied.
8: **Build** $\hat{E}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$.

With reference to the above algorithm, $\delta_{i,j}$ equals 1 if $j = i$ and 0 otherwise. From a practical point of view, one should notice that to construct matrices $\hat{E}$ and $\hat{A}$, one needs to evaluate the derivative of the transfer function. In the case that the transfer function is not available, a finite difference method can be used. However, in the specific case of TDS equipped with state-space representation (1), the derivative can be efficiently computed, as shown in Section III-B. Moreover, as a main innovative result in this paper, in the following Section III-A, some original arguments about the stability preservation between the original and the reduced-order model are provided too.

III. Approximation of stability regions

A. Results about stability approximation in $\mathcal{L}_2(i\mathbb{R})$

As the previous theoretical statements (Proposition 1 and 2) require that either the initial model be stable or that the (un)stable part be known, one needs some theoretical justifications for the use of $\mathcal{H}_2$-oriented interpolation methods in order to estimate the stability of a given system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$. These arguments are provided in the following results.

Proposition 3: For every stable system $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$, there exists a sequence of unstable systems $\mathbf{G_k} \in \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$, $k \in \mathbb{N}^*$, such that

$$\|\mathbf{H} - \mathbf{G_k}\|_{\mathcal{L}_2(i\mathbb{R})} \to 0, \quad \text{when } k \to \infty. \tag{12}$$

In other words, the set $\mathcal{H}_2(\mathbb{C}^+)$ is not an open set of $\mathcal{L}_2(i\mathbb{R})$.

Proof: Given $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$, let $\mathbf{h} \in \mathcal{H}_2(\mathbb{C}^-)$ be an element such that $\|\mathbf{h}\|_{\mathcal{L}_2(i\mathbb{R})} = 1$. The system $\mathbf{G_k} = \mathbf{H} + \frac{1}{k}\mathbf{h} \in \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$ and $\|\mathbf{H} - \mathbf{G_k}\|_{\mathcal{L}_2(i\mathbb{R})} = \frac{1}{k}\|\mathbf{h}\|_{\mathcal{L}_2(i\mathbb{R})} \to 0$ when $k \to \infty$. ■

Proposition 4: If $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$ and there exists a global minimizer $\hat{\mathbf{H}} \in \mathcal{L}_2(i\mathbb{R})$ of the $\mathcal{L}_2$ approximation problem (4), then $\hat{\mathbf{H}} \in \mathcal{H}_2(\mathbb{C}^+)$.

Similarly, if $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^-)$ and there exists a global minimizer $\hat{\mathbf{H}} \in \mathcal{L}_2(i\mathbb{R})$ of the $\mathcal{L}_2$ approximation problem (4), then $\hat{\mathbf{H}} \in \mathcal{H}_2(\mathbb{C}^-)$.

Proof: Let $\hat{\mathbf{H}} \in \mathcal{L}_2(i\mathbb{R})$ be the global minimizer of (4). Since $\mathbf{H} \in \mathcal{H}_2(\mathbb{C}^+)$, one has $\mathbf{H}^- = \mathbf{0}$. Since $\mathcal{L}_2(i\mathbb{R}) = \mathcal{H}_2(\mathbb{C}^-) \bigoplus \mathcal{H}_2(\mathbb{C}^+)$, then we have

$$\|\mathbf{H} - \hat{\mathbf{H}}\|_{\mathcal{L}_2}^2 = \|\mathbf{H}^+ - \hat{\mathbf{H}}^+\|_{\mathcal{L}_2}^2 + \|\mathbf{0} - \hat{\mathbf{H}}^-\|_{\mathcal{L}_2}^2. \tag{13}$$

Thus, $\hat{\mathbf{H}}^- = \mathbf{0}$, otherwise $\hat{\mathbf{H}}$ is not a global minimizer. ■

Proposition 5: The set of unstable systems $\mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$ is an open set of $\mathcal{L}_2(i\mathbb{R})$. In order words, given an unstable system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$, there exist an $\varepsilon > 0$ such that the ball $B_\varepsilon(\mathbf{H}) = \{\mathbf{H} \in \mathcal{L}_2(i\mathbb{R}) | \|\mathbf{G} - \mathbf{H}\|_{\mathcal{L}_2(i\mathbb{R})} < \varepsilon\} \subset \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$.

Proof: Since $\mathcal{H}_2(\mathbb{C}^+)$ is a closed set, its complement $(\mathcal{H}_2(\mathbb{C}^+))^c = \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$ is open. ■

Theorem 2 (Unstable approximated models convergence): Given an unstable system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$, there exists $n \in \mathbb{N}^*$ for which the converging sequence of approximating models G_k of dimension $k \in \mathbb{N}^*$, $k > n$, obtained from the $\mathcal{L}_2$-approximation problem (4) is also unstable.

Proof: Proposition 5 states that if a system is sufficiently close to an unstable system in the $\mathcal{L}_2(i\mathbb{R})$-norm, it is also unstable. Furthermore, the subspace of rational functions which represents the finite LTI systems is dense in $\mathcal{L}_2(i\mathbb{R})$. Hence, for a given LTI unstable system $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R}) \backslash \mathcal{H}_2(\mathbb{C}^+)$, a sequence $\mathbf{G}_k$ of systems of order k which satisfies (4), will converge to $\mathbf{H}$. Thus, due to Proposition 5, there exists an order $n \in \mathbb{N}^*$ such that if $k \geq n$, G_k will be unstable as well. ■

In other words there exist an approximation order such that if the original system is unstable, the approximated one is unstable too. Moreover, if one has found the global $\mathcal{L}_2$ minimizer of the approximation problem of order sufficiently large, it will be stable if the original model is stable due to Proposition 4 and it will be unstable if the original model is unstable, due to Theorem 2.

B. Computational (derivative) issues on TDS

Let us now move on to the specific case of linear time-invariant time-delay systems that can be represented in state-space form (1). Given a realization $\mathbf{H}$ with either state or input/output delay, in order to apply **TF-IRKA** one should both evaluate the transfer function and its derivative. Let us consider a time-delay system represented as

$$H(s) = C(sI_n - A(s))^{-1}B. \tag{14}$$

One simple example for $A(s)$ is the case where $A(s) = A_0 + A_1 e^{-s\tau}$; *i.e.* single-delay case. The derivative $H'(s)$ of $H(s)$ with respect to s is given by

$$H'(s) = -C(sI_n - A(s))^{-1}(I_n - A'(s))(sI_n - A(s))^{-1}B. \tag{15}$$

Based on these expressions, the **TF-IRKA** algorithm can be applied. Note that given an $s \in \mathbb{C}$, one has to inverse $(sI_n - A(s))$ only once in order to compute $H(s)$ and $H'(s)$. This feature is very interesting from a computational viewpoint.

C. The pure delay case

Let us consider a TDS given by the differential equation:

$$\dot{x}(t) = Ax(t - \tau) + Bu(t), \; y(t) = Cx(t) \tag{16}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$. Assume that A is diagonalizable, then there exists a nonsingular matrix

$P \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Delta = \mathbf{diag}(\lambda_1,\ldots,\lambda_n)$ such that $A = P\Delta P^{-1}$. In such a case, the transfer function of (16) reads

$$H(s) = C(sI_n - Ae^{-s\tau})^{-1}B = C(sI_n - P\Delta P^{-1}e^{-s\tau})^{-1}B$$

$$= CP^{-1}(sI_n - \Delta e^{-s\tau})PB = \sum_{k=1}^{n} \frac{c_k^T b_k}{s - \lambda_k e^{-s\tau}},$$

where b_k and c_k^T are the columns of PB and CP^{-1}, respectively. This representation is very convenient to work with as it allows one to obtain a simple expression for the derivative of the transfer function

$$H'(s) = -\sum_{k=1}^{n} c_k^T b_k \frac{1 + \tau\lambda_k e^{-\tau s}}{(s - \lambda_k e^{-s\tau})^2}. \quad (17)$$

This emphasizes that, by working with transfer functions of the form above, one can obtain the expression of the derivative without using any matrix inversion.

D. *The multiple delay case*

The formula presented in the previous section can be generalized to the multiple delay case. Let us consider, for instance, the transfer function

$$H(s) = C(sI_n + A_1 e^{-\tau_1 s} + A_2 e^{-\tau_2 s})B. \quad (18)$$

If the matrices A_1 and A_2 commute and one is diagonalizable, then they can be diagonalized in the same basis. Consequently, one obtain the following alternative formula:

$$H(s) = \sum_{k=1}^{n} \frac{c_k^T b_k}{s - \lambda_k^{(1)} e^{-s\tau_1} - \lambda_k^{(2)} e^{-s\tau_2}} \quad (19)$$

where $\lambda_k^{(i)}$ is the k-th eigenvalue of the matrix A_i. Consequently,

$$H'(s) = -\sum_{k=1}^{n} c_k^T b_k \frac{1 + \tau_1\lambda_k^{(1)} e^{-\tau_1 s} + \tau_2\lambda_k^{(2)} e^{-\tau_2 s}}{s - \lambda_k^{(1)} e^{-s\tau_1} - \lambda_k^{(2)} e^{-s\tau_2}}. \quad (20)$$

The above result can be straightforwardly generalized to the ℓ-delay case $(\tau_1,\ldots,\tau_\ell)$ provided that the corresponding matrices $A_1,\ldots,A_\ell$, commute with each other.

IV. Stability approximation of TDS

Let now apply the results and methods discussed in the previous sections in order to estimate stability regions of time-delay systems. For evaluation, we compare the results obtained with **TF-IRKA** with the ones obtained using eigenvalue-based methods; see *e.g.* [22], [23]. In what follows, we consider systems $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$ on which we apply the following procedure. First, we use the first-order $\mathcal{H}_2$ optimality conditions (6)-(7) in order to obtain a sub-optimal approximate model $\hat{\mathbf{H}} \in \mathcal{L}_2(i\mathbb{R})$ of finite order from which we derive the stability region. As suggested by Theorem 2, the instability region should be captured for any sufficiently large approximation order.

A. *Example 1 - Multiple delays and low order*

Let us consider the system $\dot{x}(t) = -x(t-\tau) - 2x(t-\gamma) + u(t),\ y(t) = x(t)$,with transfer function

$$H(s) = \frac{1}{s + e^{-\tau s} + 2e^{-\gamma s}}. \quad (21)$$

We grid the delay space $(\tau,\gamma) \in [0,2]\times[0,2]$ using a 60×60 points. For each point in this grid, a reduced order model ($r = 2$) using **TF-IRKA**, is computed. The stability region evaluated from the reduced model is plotted in Figure 1.

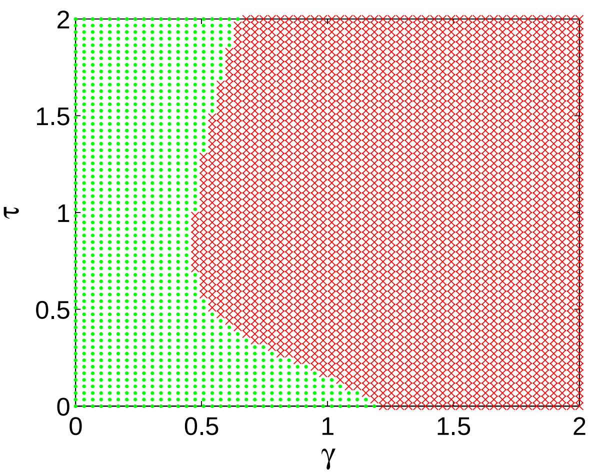

Fig. 1. Example 1: System governed by (21). Red crosses (unstable) and green dots (stable). All points interpolated with a model of order $r = 2$ using **TF-IRKA**.

B. *Example 2 - Feedback delay and controller gain*

Let us consider now the system used in [24], [25], governed by $\dot{x}(t) = Ax(t) + Bu(t)\ y(t) = Cx(t)$, where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T. \quad (22)$$

We add to this model the delayed static output feedback $u(t) = -ky(t) + y(t-\tau)$. The delay acts here as a control parameter and serves to approximate a derivative action. The resulting closed-loop model is given by $\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau)$ where $A_0 = A - BCk$ and $A_1 = BCk$. In order to apply the proposed method, we add to the above model a virtual input B_v and a virtual output C_v as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + B_v u(t) \\ y(t) &= C_v x(t) \end{aligned} \quad (23)$$

where $B_v = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$ and $C_v = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. We consider now the parameter space $(k,\tau) \in [0,6]\times[0,6]$. The gridding is performed as follows: we consider 150 equidistant grid points for the delay τ and 150 logarithmically spaced grid points for the controller gain k. We then apply, for each of these points, the **TF-IRKA** algorithm, from which we obtain a model of order $r = 6$. The obtained stability region is depicted in Figure 2. By comparing the results with the one obtained in [25], both stability regions appear very similar. The red points in the bottom are due to the fact the algorithm did not archive global minimizer.

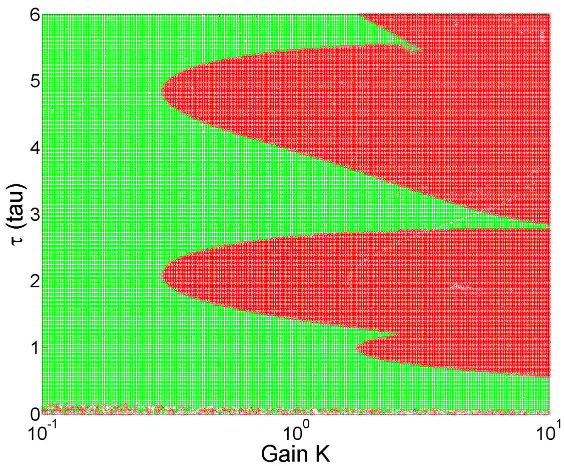

Fig. 2. Example 2: System governed by (23). Red crosses (unstable) and green dots (stable). All points interpolated by a model of order $r = 6$.

C. Example 3 - Single delay (in feedback) large-scale system

Let us now consider the clamped beam model from COMP$l_e ib$ library [26] whose order is $n = 348$, denoted $\mathbf{H}_{BEAM}$. For this system, a PID controller has been designed without taking into account any delay. The transfer function of this PID is given by $H_{PID}(s) = \frac{9.791s^2+0.04095s+0.0.07712}{s^2+0.0628s}$. Therefore the closed-loop system including the controller, denoted $\mathbf{H}_{PID}$, in series has dimension $n = 350$. We now add a constant delay $\tau \in [0, 10]$ between the output of the system and the input of the controller, and a scaling parameter $k \in [0, 1.5]$ on the gains of the PID controller. The resulting transfer function is then given by

$$H_{\text{feedback}}(s) = \frac{kH_{\text{PID}}(s)H_{\text{BEAM}}(s)}{1 + kH_{\text{PID}}(s)H_{\text{BEAM}}(s)e^{-\tau s}}. \quad (24)$$

As before, we consider several values for τ and k, we compute a reduced order model of order $r = 14$ and we evaluate the stability of the reduced model. The results are depicted in Figure 3 (right). For comparison, the exact stability regions obtained using the Nyquist criterion are reported in Figure 3 (left).

Reader should note that all unstable pairs (k_l, τ_j) from Figure 3 are well captured. This remark can be seen as consequence of Proposition 4 and 5, while unstable points on the bottom left corner correspond to points where the reduced system is not global minimizer and, even it is sufficiently closed to a stable system, it can be unstable due to Proposition 3. This example demonstrates that the proposed methodology is useful when considering large-scale time-delay systems.

D. Example 4 - Multiple delays (in feedback) large-scale system

Finally, let us consider the industrial MIMO aircraft model given in [5] whose order is $n \approx 600$ and with $n_u = 2$ and $n_p = 3$. For this model, an anti-vibration controller $\mathbf{K}$ has been designed without taking into account potential output delays τ_1, τ_2 and τ_3 in the feedback loop (see Figure 4).

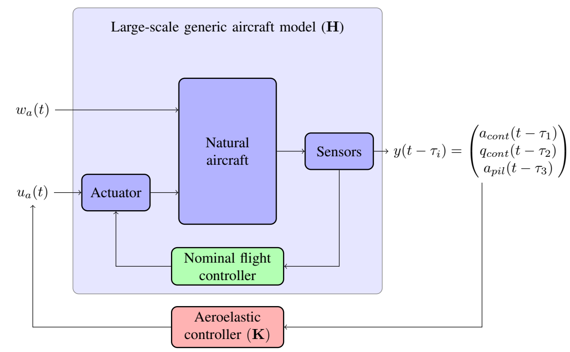

Fig. 4. A generic aircraft LTI block diagram representation with nominal and aeroelastic controllers, interconnected with delays in the loop.

We consider a fix $\tau_1 = 17$ms and $\tau_2, \tau_3 \in [0, 17]$ms. Following the proposed approximation-based algorithm, the stability regions are reported in Figure 5 with a reduced model of order $r = 6$. Note that, due to the large dimension of the original system, standard approaches relying on linear matrix inequalities cannot be used. The tractability of the proposed approach is a clear advantage. Moreover, in this case, the Nyquist approach is difficult to apply due to the interplay between the different delays.

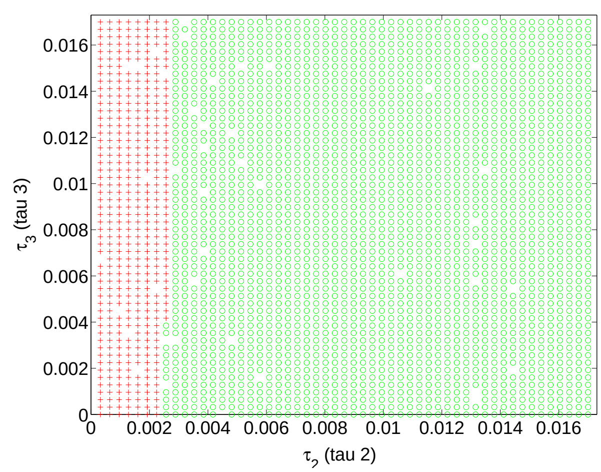

Fig. 5. Example 5: System represented in Figure 5. Red crosses (unstable) and green dots (stable). All points interpolated by a model of order $r = 6$. Points where algorithm did not converges are left blank.

V. Conclusions and Future Works

In this paper, the problem of estimating stability regions of large scale TDS is addressed by using optimal model reduction techniques. First, specific cases to compute TDS

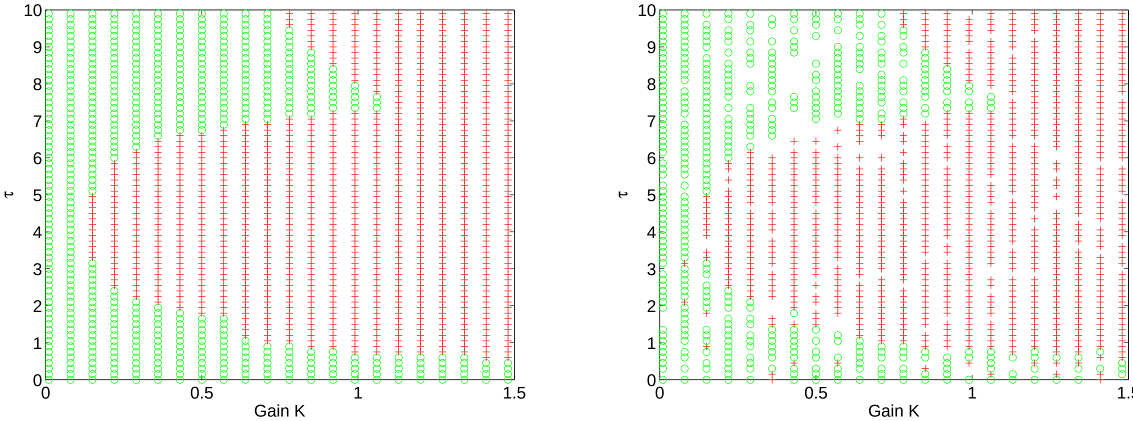

Fig. 3. Example 4: System whose transfer function is given by (24). Red (unstable) and green (stable). Points obtained applying Nyquist technique (left frame). Points interpolated by a model of order $r = 14$ obtained with **TF-IRKA** (right frame). Points where algorithm didn't converges are let blank.

derivative functions in the large-scale framework have been proposed, allowing to enhance both algorithm accuracy and velocity. Moreover, when the transfer function lives in $\mathcal{H}_2(\mathbb{C}^+)$ or $\mathcal{H}_2(\mathbb{C}^-)$, this method has been shown to achieve the first order optimality conditions. However, if $\mathbf{H} \in \mathcal{L}_2(i\mathbb{R})$ sub-optimal conditions show to provide good approximation. In addition, as stated in Theorem 2, a necessary condition to assess instability, provided that approximated order is sufficiently high, has been provided, which is a new result. The overall approach have been validated on many different TDS (single, multiple delays) and on both academic and industrial very large-scale models providing promising perspectives for the stability estimation of LS TDS.

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